Adiabatic Assumption for Wave Propagation

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The common assumption that the speed of sound in a gas is determined by the adiabatic compressibility because the pressure variations are so rapid that no heat flow can take place is shown to be incorrect. The criterion for applicability of the adiabatic assumption in a free wave is rather that the frequency be lower than a certain very high limit. The propagation of sound in tubes is also discussed and it is shown that the frequency must, in addition, lie above a certain low limit if the adiabatic assumption is to be valid.

All introductory courses in the mechanics of waves include a discussion of the propagation of plane sound waves through gases. The two fundamental equations are Newton's second law of motion, relating acceleration to pressure gradient, and the continuity equation expressing the conservation of mass. Both of these can be linearized by the assumption that the acoustic pressure in the wave is infinitesimal in comparison with the static pressure.

A third equation is needed to complete the set and this is the equation of state relating pressure and density in the acoustic wave. The adiabatic gas equation

$$pV^{\gamma}=\text{constant} \tag{1}$$

is known to be appropriate here, and it is an almost universal practice to justify this assumption in words somewhat as follows:

> The alternations of pressure and density are so rapid that...no heat energy has time to flow away from the compressed part of the gas before this part is no longer compressed.

Or perhaps:

> The changes in state normally occur so rapidly in a sound wave that there is no time for the temperature to equalize.... The changes of state are not isothermal, therefore, but adiabatic.

These two quotations are taken respectively from the classic treatment by Morse[1] and from an excellent account published recently by Meyer and Neumann.[2]

According to Lord Rayleigh,[3] Newton made the opposite assumption that the compressions and rarefactions occur isothermally, and it was Laplace who explained the discrepancy between theory and experiment by introducing the adiabatic assumption, justifying this in words very similar to those quoted above. Rayleigh quotes a discussion given later by Stokes which applies, however, to the propagation of sound waves in gases at very high temperatures where radiation is the dominant mechanism for heat transfer.

The rather surprising fact is actually that these justifications are completely incorrect, and the reason for the validity of the adiabatic assumption for sound waves is that the changes in pressure, and hence in temperature, occur so slowly that appreciable heat flow cannot take place. This fact is actually "well known" in a scientific sense but is certainly not appreciated in a more elementary sense by the writers of textbooks and monographs.

To see how the apparently paradoxical statement which I have made above comes to be true, let us consider the first law of thermodynamics applied to a small volume of gas through which a plane wave is moving in the x direction. The relevant equation for a small element of gas during a short time interval dt is

$$dQ=\rho_0 C_v dT+p_0 dV, \tag{2}$$

where ρ_0 and p_0 are the static density and pressure, respectively; C_v is the specific heat at constant volume; and dQ, dT, and dV are the changes in heat content, temperature, and volume, as usual.

The essence of the adiabatic assumption is to set the left hand side of Eq. (2) (which represents the change in heat content of the gas element) equal to zero. We then have

$$\rho_0 C_v dT=-p_0 dV, \tag{3}$$

and Eq. (1) follows in the usual way.[4] For this to be a valid approximation, however, the term dQ, which appears in Eq. (2) essentially as a correction to Eq. (3), must be negligibly small in comparison with the terms on either side of Eq. (3). Writing this condition as

$$| dQ | \ll | \rho_0 C_v dT | \tag{4}$$

and dividing by the time interval dt gives

$$| \partial Q/\partial t | \ll \rho_0 C_v | \partial T/\partial t |. \tag{5}$$

Now the change with time of the heat content Q is caused principally by conduction in the case of gases at ordinary temperatures, so that we can write

$$\partial Q/\partial t=K(\partial^2 T/\partial x^2), \tag{6}$$

where K is the thermal conductivity of the gas. The inequality (5) therefore becomes

$$K | \partial^2 T/\partial x^2 | \ll \rho_0 C_v | \partial T/\partial t | \tag{7}$$

or, introducing the thermal diffusivity κ defined by $\kappa=K/\rho_0 C_v$,

$$| \partial^2 T/\partial x^2 | \ll \kappa^{-1} | \partial T/\partial t |. \tag{8}$$

Now suppose we have a sound wave with angular frequency ω, propagation constant k, and amplitude A moving in the x direction, so that the acoustic pressure varies as $A \sin(kx-\omega t)$. Because the equation of state,

$$pV=RT, \tag{9}$$

is linear for waves of small amplitude, the temperature T also varies to first order as $T_0+B \sin(kx-\omega t)$, where T_0 is the static temperature and B is proportional to the amplitude A. The condition (8) can then be written

$$k^2 | \sin(kx-\omega t) | \ll (\omega/\kappa) | \cos(kx-\omega t) |, \tag{10}$$

which requires that

$$k^2 \ll \omega/\kappa \tag{11}$$

or

$$\omega \ll c^2/\kappa, \tag{12}$$

where c is the sound velocity defined by $c=\omega/k$. It follows, therefore, from inequality (12), that the condition for validity of the adiabatic assumption is that the frequency of the sound wave should be low!

To make clearer the reason for this situation, let us consider the sound wave in simple pictorial terms. The regions of high and low pressure (and hence of high and low temperature) are separated by a distance $\lambda/2$, and the condition for validity of the adiabatic assumption is that negligible heat flow occur between them in the course of a half-period of the wave during which time the temperature distribution will be reversed. For a wave of angular frequency ω, this time is π/ω.

Now heat flow is a diffusive phenomenon and, in rough terms, the distance over which heat diffuses in time t is $(\kappa t)^{1/2}$ where κ is the thermal diffusivity. The condition for validity of the adiabatic assumption is therefore

$$(\kappa\pi/\omega)^{1/2} \ll \lambda/2, \tag{13}$$

which, since $\lambda=2\pi c/\omega$, reduces to

$$\omega \ll \pi c^2/\kappa, \tag{14}$$

which is essentially equivalent to relation (12).

It is interesting to insert some numbers into this criterion to determine the range of validity of the adiabatic assumption. For air, $c \simeq 340$

m/sec and $\kappa \simeq 10^{-5}$ m^2/sec^{-1}, so that we require $\omega \ll 10^{10}$ sec^{-1}. This criterion is amply met for all acoustic waves. The conduction of heat does, however, represent a dissipation mechanism so that there is an attenuation during propagation which increases as the frequency increases. The quantity leading to the criterion (12) is essentially the fraction of heat energy lost per wavelength, and this must be multiplied by the wave number to find the attenuation coefficient, which thus increases more than linearly with increasing frequency and actually as ω^2. The attenuation of sound in air becomes large at frequencies of order 10^5 Hz, but this is due to viscous and molecular-excitation losses as well as to heat conduction.

As a related case it is interesting to apply the same sort of reasoning to discuss the propagation of a sound wave in a tube. Clearly, if the frequency is low enough, the air will be able to exchange heat with the tube walls so that the behavior will be isothermal rather than adiabatic and the wave velocity will be correspondingly lower. The discussion follows that leading to (13), except that the relevant distance for heat transfer is now not a half-wavelength for the wave itself but rather the radius R of the tube. For the adiabatic approximation to be valid in a heavy tube with conducting walls we must therefore have

$$(\kappa\pi/\omega)^{1/2} \ll R \tag{15}$$

or

$$\omega \gg \pi\kappa/R^2. \tag{16}$$

Note that this inequality is in the opposite sense to (14). For a tube of radius 1 cm this transition frequency is of order 10^{-1} Hz, but for capillary tubes it lies in the audio frequency range. This transition and the related change in sound velocity were observed long ago by Kundt,[5] and it is probably from confusion between this and the free-wave situation that the textbook misunderstandings have arisen. For the adiabatic approximation to be valid for tubes, of course, both conditions (12) and (16) must be satisfied.

This curiosity came to my attention during the course of a research program on musical acoustics, supported by the Australian Research Grants Committee.

[1] P. M. Morse, *Vibration and Sound* (McGraw-Hill, New York, 1948), p. 221.

[2] E. Meyer and E. G. Neumann, *Physical and Applied Acoustics* (Academic, New York, 1972), p. 3.

[3] J. W. S. Rayleigh, *The Theory of Sound* (McMillan, London, 1896; Dover, New York, 1945), Vol. 2, pp. 18–29.

[4] See, for example, R. Resnick and D. Halliday, *Physics* (Wiley, New York, 1966), pp. 586–587.

[5] Reference 3, pp. 57–59.