Quantum Mechanics part 2,
Assignment - 2007

Part 1. Relativistic equations

I. Spinless particle, Schrödinger equation

Consider a nonrelativistic particle, which is described by the conventional nonrelativistic Schrödinger equation, which for a free particle has the known form

$$E\psi = \frac{\mathbf{p}^2}{2m}\psi \tag{1.1}$$

Here $\mathbf{p} = -i\nabla$ is the operator of momentum. Let the particle has a charge e and zero spin.

- Using gauge invariance write down the Schrödinger equation in a static external electromagnetic field described by potentials V, **A**.

Consider a particular case of a static magnetic field **B,** choosing the gauge

$$V = 0, \quad A^\mu = (0, \mathbf{A}) \tag{1.2}$$

- Write down the Schrödinger equation for this case.

Consider a specific case of a homogeneous magnetic field choosing its direction as the z-axis

$$\mathbf{B} = (0, 0, B) \tag{1.3}$$

- Verify that the potential
  $$\mathbf{A} = (-By, 0, 0) \tag{1.4}$$
  complies with Eq.(1.3), i.e. that
  $$\nabla \times \mathbf{A} = \mathbf{B} \tag{1.5}$$
  Write down the Schrödinger equation in the chosen gauge.

The Hamiltonian in this equation does not depend on x, z , which allows one to look for a solution in the form

$$\psi(\mathbf{r}) = \exp\left[i\left(p_x x + p_z z\right)\right]\chi(y) \tag{1.6}$$

- Substituting Eq.(1.6) in the Schrödinger equation demonstrate that $\chi(y)$ satisfies
  $$\left(E - \frac{p_z{}^2}{2m}\right)\chi(y) = \left(\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2 (y - y_0)^2}{2}\right)\chi(y) \tag{1.7}$$
  where p_z is the value of the momentum in the z-direction (which is conserved), while $\hat{p}_y$ is a conventional differential operator $\hat{p}_y = -i\dfrac{\partial}{\partial y}$, y_0 is a constant (find it), and
  $$\omega = \frac{eB}{m} \tag{1.8}$$
  is the cyclotron frequency.

- Show that Eq.(1.7) leads to the Landau spectrum

$$E = E_{n,p_z} = \left(n + \frac{1}{2} \right) \omega + \frac{p_z^{\,2}}{2m}, \quad n = 0,1,\ldots \tag{1.9}$$

II. Spin 1/2, Pauli equation

Consider an electron in the homogeneous magnetic field, describing it by the Pauli equation. Remember that the Pauli Hamiltonian H_P differs from the Schrödinger Hamiltonian H_S by an additional term δU which takes into account an interaction of the magnetic dipole moment with the field

$$\begin{aligned} &H_P = H_S + \delta U \\ &\delta U = -\boldsymbol{\mu} \cdot \mathbf{B} = -\mu \boldsymbol{\sigma} \cdot \mathbf{B} \\ &\mu = \frac{e}{2m} = -\frac{|e|}{2m} \end{aligned} \tag{2.1}$$

- Using the spectrum of the spinless particle Eq.(1.9) and Eq.(2.1) demonstrate that the Landau spectrum of the electron in the magnetic field has the following form

  $$E = E_{n,p_z,s} = \left(n + \frac{1}{2} + s \right) \omega + \frac{p_z^{\,2}}{2m}, \quad n = 0,1,\ldots \tag{2.2}$$

  where $s = \pm 1/2$ is the projection of the spin on the direction of the magnetic field.

III. Scalar particle, Klein-Gordon equation

- Write down the Klein-Gordon equation for a free particle. Using gauge invariance generalize it for the case of a particle in an arbitrary external electromagnetic field.

- Consider a particular case of a homogeneous static magnetic field. Write down the Klein-Gordon equation explicitly using the gauge Eq.(1.2).
  Hint. Take into account that for a static field

  $$\phi(\mathbf{r},t) = \exp\left(-i\,\varepsilon\, t\right)\, \varphi(\mathbf{r}) \tag{3.1}$$

  depends on time trivially
  Verify that the Klein-Gordon equation in the case considered looks similar to the Schroedinger equation. Using this similarity demonstrate that the Landau spectrum in this case reads

  $$\varepsilon = \varepsilon_{n,p_z} = \left[2\left(n + \frac{1}{2} \right) |e|\, B + p_z^{\,2} + m^2 \right]^{1/2} \tag{3.2}$$

- Write down conditions on $B,\ p_z$, under which the relativistic spectrum Eq.(3.2) can be described by the nonrelativistic Eq.(1.9).

IV. Electron, Dirac equation.

- Write down the Dirac equation for the free electron
- Write down the Dirac equation for the electron in an external electromagnetic field.

- Show that the later equation can be presented as the second-order differential equation

$$\left(p_\mu - eA_\mu\right)\left(p^\mu - eA^\mu\right)\psi - \frac{i\,e}{2}F_{\mu\nu}\sigma^{\mu\nu}\psi = m^2\,\psi \tag{4.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\sigma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{4.2}$$

- Verify the following identity

$$\frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu} = \boldsymbol{\alpha}\cdot\mathbf{E} + i\,\boldsymbol{\Sigma}\cdot\mathbf{B} \tag{4.3}$$

where in the standard representation the Dirac matrixes read

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{4.4}$$

- Apply Eq.(4.1) for the case of a homogeneous static magnetic field. Remember that in this case

$$\psi(\mathbf{r},t) = \exp(-i\,\varepsilon t)\psi(\mathbf{r}) \tag{4.5}$$

- Show that the results for the Pauli equation and the Klein-Gordon equation provide an easy way to find the Landau spectrum in the case at hand

$$\varepsilon = \varepsilon_{n,p_z,S} = \left[2\left(n + \frac{1}{2} + s\right)|\,e\,|\,B + p_z{}^2 + m^2\right]^{1/2} \tag{4.6}$$

Hint: remember that

$$\Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.7}$$

- Show that in the nonrelativistic limit Eq. (4.6) is reduced to (2.2).

Part 2. Scattering theory

Consider propagation of a (nonrelativistic) particle in the s-wave ($l = 0$) in an attractive potential

$$U(\mathbf{r}) = \begin{cases} -U < 0, & r < a \\ 0, & r > a \end{cases} \tag{1.1}$$

- Solve the radial Schrödinger equation for the s-wave, presuming that

$$R_{K,0} = \begin{cases} \dfrac{2}{r}\sin(Kr + \delta_0), & r > a \\ const, & r = 0 \end{cases} \tag{1.2}$$

- Find an explicit equation, which defines the scattering phase δ_0 .
- Find conditions on the potential, which lead to a resonance at low energies. Remember that the low-energy resonance manifests itself through the following behaviour of the phase

$$\cot\,\delta_0 = -\frac{\kappa}{K}, \qquad |\kappa|,\, K \ll \frac{1}{a} \tag{1.3}$$