Scattering

1. Wave function at large distances

$$\psi \underset{r\to\infty}{=} e^{i\vec{k}\cdot\vec{r}} + f(\theta)\,\frac{e^{ikr}}{r} \qquad (1)$$

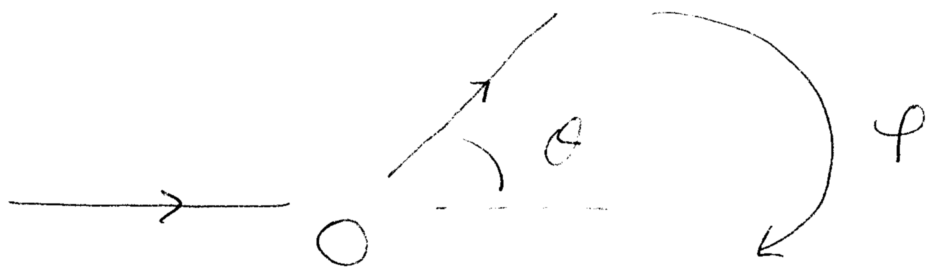
i.e. $kr \gg 1$

$r \gg a \leftarrow$ "size" of the scatterer

$f(\theta)$ - scattering amplitude

$$H = \frac{p^2}{2m} + U(\vec{r}) \qquad (2)$$

$$U(\vec{r}) = U(r) \qquad (3)$$



(4)

(3) $\Rightarrow$ $f = f(\theta)$ does not depend on φ

Flux of probability (particles)

$$\psi = \psi(\vec{r}, t) \qquad (5)$$

$$\rho(\vec{r}, t) = |\psi|^2 \qquad (6)$$

$$\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0 \qquad (7)$$

(7) =>

$$\vec{j} = \frac{1}{2im}\left(\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi\right) \qquad (8)$$

$$\begin{cases} \text{If} \quad \psi = \psi_{in} = e^{i\vec{k}\vec{r}} \\ \text{then} \quad \vec{j}_{in} = \dfrac{\vec{k}}{m} = \vec{v} \end{cases} \tag{9}$$

$$\begin{cases} \text{If} \quad \psi = \psi_{out} = f\,\dfrac{e^{ikr}}{r} \\ \text{then} \quad \vec{j}_{out} = |f|^2 \dfrac{\vec{n}}{r^2}\, v \end{cases} \tag{10}$$

$$\frac{j_{out}}{j_{in}} = |f|^2 \frac{1}{r^2} \tag{11}$$

$$\frac{j^{out}\, dS}{j^{in}} \underset{(11)}{=} |f|^2 \frac{r^2 d\Omega}{r^2} = \tag{12}$$

$$= |f(\theta)|^2 d\Omega = d\sigma$$

$$\frac{d\sigma}{d\Omega}(\theta) \underset{(12)}{=} |f(\theta)|^2 \tag{13}$$

$$\sigma_{tot} \underset{(13)}{=} \int |f(\theta)|^2 d\Omega =$$

$$= 2\pi \int_0^\pi |f(\theta)|^2 \sin\theta \, d\theta \tag{14}$$

Theorem:

If

$$\Delta \psi + k^2 \psi = Q \qquad (15)$$

then

$$\psi = -\frac{1}{4\pi} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} Q(\vec{r}')\, d^3r' + \psi_0$$

Particular know case:

for $k=0$

$$\Delta \psi = Q \qquad (16)$$

$$\psi = -\frac{1}{4\pi} \int \frac{Q(r')}{|\vec{r}-\vec{r}'|} d$$

Proof: Begin proof

$$(\Delta + k^2) \frac{e^{ikr}}{r} = \frac{1}{r}\left(\frac{d^2}{dr^2} + k^2\right) e^{ikr} = \qquad (1$$

$$= 0 \quad \text{when} \quad r > 0$$

(15) follows from (16, 17)

End of proof

$(2) \Rightarrow$

$$(\Delta + k^2)\psi = -2mU\psi \qquad (18$$

$$\psi \underset{(15,18)}{=} \underbrace{\psi_0^{(0)}}_{e^{i\vec{k}\cdot\vec{r}}} - \frac{m}{2\pi}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} U(r') \cdot \psi(r')\, d^3r \qquad (19)$$

$$\frac{1}{|r-r'|} \underset{r\to\infty}{\longrightarrow} \frac{1}{r} \qquad (20)$$

$$\left\{ \begin{array}{l} |\vec{r}-\vec{r}'| \underset{r\to\infty}{\longrightarrow} r - \vec{n}\cdot\vec{r}' \qquad \vec{n} = \frac{\vec{r}}{r} \\ \exp\left(ik|\vec{r}-\vec{r}'|\right) \underset{r\to\infty}{\simeq} e^{ikr - i\vec{k}'\cdot\vec{r}'} \qquad (21 \\ \qquad \vec{k}' = k\vec{n} \end{array} \right.$$

$$\psi \xrightarrow[r\to\infty]{(20,21)} e^{i\vec{k}\vec{r}} + f(\theta)\,\frac{e^{ikr}}{r} \qquad (22)$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}'\vec{r}'}\,U(\vec{r}')\,\psi_{\vec{k}}(\vec{r}')\,d^3r'$$

$$= -\frac{m}{2\pi\hbar^2}\,\langle \vec{k}'\,|\,U\,|\,\psi_{\vec{k}}\rangle \qquad (23)$$