Partial waves analyses
Scattering phases

$$\psi_{\vec{k}}(\vec{r}) = \sum_{\ell=0}^{\infty} A_\ell \, P_\ell(\cos\theta) \, R_{k\ell}(r) \tag{58}$$

$$\left(\Delta + k^2 - 2mU \right) \psi_{\vec{k}} = 0 \tag{59}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} R_{k\ell} \right) + \left(k^2 - \frac{\ell(\ell+1)}{r^2} - 2mU \right) R_{k\ell} = 0 \tag{60}$$

$$\left\{ \begin{aligned} \Delta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{\Lambda}{r^2} \\ \Lambda &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{\partial^2}{\partial\varphi^2} \\ \Lambda P_\ell &= -\ell(\ell+1) \end{aligned} \right. \tag{61}$$

Plane wave

$$\underbrace{e^{i\vec{k}\vec{r}}}_{\psi^{(0)}} = \sum_{\ell=0}^{\infty} A_\ell^{(0)} P_\ell(\cos\theta)\, R_{k\ell}^{(0)}(r) \tag{62}$$

$$A_\ell^{(0)} = \frac{1}{2k}(2\ell+1)\, i^\ell \tag{63}$$

$$R_{k\ell}^{(0)}(r) = \sqrt{\frac{2\pi k}{r}}\, J_{\ell+\frac{1}{2}}(kr) \tag{64}$$

Bessel's functions

$$\begin{cases} \ell = 0 \\ R_{k0}^{(0)}(r) = \dfrac{2}{r}\sin kr \end{cases} \tag{65}$$

$$R_{k\ell}^{(0)}(r) \xrightarrow[r\to\infty]{} \frac{2}{r}\sin\left(kr - \frac{\ell\pi}{2}\right) \tag{66}$$

$$R_{k\ell}^{(0)}(r) \xrightarrow[r\to 0]{} C_{k\ell}^{(0)}\, r^\ell \tag{67}$$

$$R_{k\ell}(r) \underset{r\to 0}{\longrightarrow} C_{k\ell}\, r^{\ell} \qquad (68)$$

$$R_{k\ell}(r) \underset{r\to\infty}{\longrightarrow} \frac{2}{r} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) \qquad (69)$$

Theorem:

$$A_\ell = \frac{1}{2k}(2\ell+1)\, i^{\ell} e^{i\delta_\ell} \qquad (70)$$

Proove

$$(69) \Rightarrow \quad R_{k\ell} \simeq \frac{1}{ir}\left[(-i)^{\ell} e^{i(kr+\delta_\ell)} - i^{\ell} e^{-i(kr+\delta_\ell)}\right] \qquad (71)$$

$$\psi \underset{(58,70,71)}{\simeq} \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\theta) \cdot \left[(-)^{\ell+1} e^{-ikr} + e^{2i\delta_\ell} e^{ikr}\right] \qquad (72)$$

$$\psi^{(0)} \simeq \frac{1}{2ikr} \sum (2l+1)\, P_l(\cos\theta) \left[(-)^{l+1} e^{-ikr} + e^{ikr} \right] \qquad (73)$$

$$\psi \underset{(71,73)}{\simeq} \underbrace{\psi^{(0)}}_{e^{i\vec{k}\vec{r}}} + \sum (2l+1)\, P_l(\cos\theta)\, \frac{e^{2i\delta}-1}{2ik}\, \frac{e^{ikr}}{r} \qquad (74)$$

<u>(74) proves (70)</u>

$$= e^{i\vec{k}\vec{r}} + f(\theta)\, \frac{e^{ikr}}{r} \qquad (75)$$

$$f(\theta) \underset{(74)}{=} \underline{\frac{1}{k} \sum (2l+1)\, P_l(\cos\theta)\, \frac{e^{2i\delta}-1}{2i}} \qquad (76)$$

$$f_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) =$$

$$= \frac{1}{k} e^{i\delta_\ell} \sin\delta_\ell \qquad (77)$$

$$f(\theta) = \sum_\ell (2\ell+1) f_\ell \, P_\ell(\cos\theta) \qquad (78)$$

$$\sigma_\ell = 4\pi\,(2\ell+1)\,|f_\ell|^2 = \qquad (79)$$

$$= \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$$

$$\left\{\begin{aligned} \sigma &= \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell && (80) \\ &\underset{(79)}{=} \sum \sigma_\ell && (81) \end{aligned}\right.$$