Klein - Gordon equation

Schrödinger and Fock are said to know this equation

Units :

$$\hbar = c = 1$$

$$\frac{e^2}{\hbar c} \equiv e^2 = \alpha = \frac{1}{137.036}$$

$$\mathcal{E}, \vec{p} \tag{1}$$

Lorentz invariance needs that

$$p^\mu = (\mathcal{E}, \vec{p}) \tag{2}$$

$$p_\mu p^\mu = \mathcal{E}^2 - \vec{p}^{\,2} = \text{Const} \equiv C \tag{3}$$

$$\mathcal{E} = \sqrt{C + \vec{p}^{\,2}} \tag{4}$$

$$\mathcal{E} \underset{p \to 0}{\to} \sqrt{C} + \frac{p^2}{2\sqrt{C}} \tag{5}$$

$$(5) \Rightarrow \sqrt{C} = m - \text{mass} \tag{6}$$

$$\mathcal{E} \underset{(4,6)}{=} \sqrt{m^2 + \vec{p}^{\,2}} \tag{7}$$

$$(7) \Rightarrow \quad \underset{\vec{p}=0}{\mathcal{E}} = mc^2 \qquad (8)$$

(8) follows directly from Loretz invariance (2)

$$(3) \Rightarrow \quad \mathcal{E}^2 - \vec{p}^{\,2} = m^2 \qquad (9)$$

Take a plane wave

$$\varphi = e^{i(\vec{p}\cdot\vec{r} - \vec{\mathcal{E}}t)} \qquad (10)$$

Introduce operators

$$\hat{\mathcal{E}} = i\frac{\partial}{\partial t} \qquad \hat{\vec{p}} = -i\vec{\nabla} \qquad (11)$$

$(9-11) \Rightarrow$

$$\left(\hat{\varepsilon}^2 - \hat{\vec{p}}^{\,2}\right)\varphi = m^2\varphi \qquad (12)$$

Other forms

$$\left\{\begin{aligned}
&\left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\varphi = m^2\varphi \\
&\hat{p}_\mu \hat{p}^\mu \varphi = m^2\varphi \\
&\qquad \hat{p}_\mu = i\frac{\partial}{\partial x^\mu} = i\partial_\mu \\
&\hat{p}^2\varphi = m^2\varphi \\
&-\partial^2\varphi = m^2\varphi
\end{aligned}\right. \qquad (13)$$

Gauge invariance

Quantum theory is invariant under transformations

$$\psi \to \psi' = e^{-i\alpha}\psi \qquad (14)$$

where α is a constant $\qquad (15)$

(14, 15) is called global gauge transformation

Local gauge transformation

$$\left\{\begin{aligned} &\psi \to \psi' = e^{-i\alpha}\psi \qquad \alpha = \alpha(x) \\ &A_\mu \to A_\mu' = A_\mu + e^{-1}\partial_\mu \alpha \end{aligned}\right. \qquad (16)$$

$$\nabla_\mu = \partial_\mu + ieA_\mu \tag{17}$$

$$\nabla_\mu \psi \to \nabla'_\mu \psi' = \left(\partial_\mu + ieA'_\mu\right)\psi' =$$

$$= \left(\partial_\mu + ieA_\mu + i\,\partial_\mu \alpha\right) e^{-i\alpha}\psi$$

$$= e^{-i\alpha}\left(\underbrace{\partial_\mu + ieA_\mu}_{\nabla_\mu} + \cancel{i\partial_\mu \alpha} - \cancel{i\partial_\mu \alpha}\right)\psi \tag{18}$$

$$= e^{-i\alpha}\,\nabla_\mu \psi$$

(16, 18) $\Longrightarrow$

$$\begin{cases} \psi \to \psi' = e^{-i\alpha}\psi \\ \nabla_\mu \psi \to \nabla_\mu' \psi' = e^{-i\alpha}\nabla_\mu \psi \end{cases} \qquad (19)$$

$$A_\mu' = A_\mu + \frac{1}{e}\partial_\mu \alpha$$

The rule (law) (20)

Nature alway uses covariant derivative, or $\underline{p_\mu - eA_\mu}$

(21)

(19), (20) $\Longrightarrow$ QED is gauge inv

(because all phases in (19) are cancelled out in probabilities)

Klein-Gordon equation in external EM field

$$p_\mu - eA_\mu \tag{21}$$

$$\begin{cases} p_\mu = (\hat{\varepsilon}, -\hat{\vec{p}}) \\ p^\mu = (\hat{\varepsilon}, \hat{\vec{p}}) \end{cases} \qquad \hat{\varepsilon} = i\frac{\partial}{\partial t}$$

$$\vec{p} = -i\vec{\nabla}$$

$$\begin{cases} A_\mu = (A_0, -\vec{A}) \\ A^\mu = (A_0, +\vec{A}) \end{cases}$$

$$p_\mu - eA_\mu = \left(\varepsilon - eA_0, -(\vec{p} - e\vec{A})\right)$$

$$p^\mu - eA^\mu = \left(\varepsilon - eA_0, \vec{p} - e\vec{A}\right)$$

$$\hat{p}_\mu \hat{p}^\mu \varphi \overset{(13)}{=} m^2 \varphi$$

$$\Downarrow$$

$$(p_\mu - eA_\mu)(p^\mu - eA^\mu)\varphi = m^2 \varphi \qquad (14)$$

$$\Downarrow$$

$$\left[(\hat{p}_0 - eA_0)^2 - (\hat{\vec{p}} - e\vec{A})^2\right]\varphi = m^2\varphi \qquad (15)$$

Same in different notation

$$-\partial_\mu \partial^\mu \varphi = m^2 \varphi \qquad (16)$$

$$\Downarrow$$

$$-\nabla_\mu \nabla^\mu \varphi = m^2 \varphi \qquad (17)$$

(14, 15, 17) is the Klein-Gordon equation in different notations

Coulomb field

$$V = A_0 = -\frac{Ze^2}{r} \qquad (18)$$

Units $\quad \frac{e^2}{\hbar c} = \alpha$

Coulomb force $\quad F = \frac{q_1 q_2}{r}$ $\qquad (19)$

Gauss' law $\quad \vec{\nabla} \cdot \vec{E} = 4\pi\rho$

$$A^\mu = (V, \vec{0}) \qquad (20)$$

$$\varphi(\vec{r}, t) = e^{-i\varepsilon t}\varphi(\vec{r}) \qquad (21)$$

Static potential (18)
makes possible
static solution $\varphi(\vec{r})$

(15), (20) $\Rightarrow$

$$\left[\left(i \frac{\partial}{\partial t} - V \right)^2 + \Delta \right] \varphi(\vec{r}, t) = m^2 \varphi(\vec{r}, t) \qquad (22)$$

(21, 22) $\Rightarrow$

$$\left[(\varepsilon - V)^2 + \Delta \right] \varphi(\vec{r}) = m^2 \varphi(\vec{r}) \qquad (23)$$

(23, 18)

$$\left(\varepsilon + \frac{Ze^2}{r} \right)^2 \varphi = \left(-\Delta + m^2 \right) \varphi \qquad (24)$$

$$\varphi(\vec{r}) = \varphi(r)\, Y_{\ell m} \qquad (25)$$

ℓ-th wave

$$\Delta \varphi(\vec{r}) \underset{(25)}{=} \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\varphi(r)}{dr}\right) - \frac{\ell(\ell+1)}{r^2}\varphi\right] \cdot Y_{\ell m} \qquad (26)$$

$(24, 26) \Rightarrow$

$$\left(\varepsilon^2 + \frac{2\varepsilon Z e^2}{r} + \frac{(Ze^2)^2}{r^2}\right)\varphi =$$

$$= -\left(\varphi'' + \frac{2}{r}\varphi'\right) + \left(\frac{\ell(\ell+1)}{r^2} + m^2\right)\varphi \qquad (27)$$

$$(\varepsilon^2 - m^2)\,\varphi \underset{(27)}{=} -\Delta_r\varphi - \frac{2\varepsilon Z e^2}{r}\varphi + \frac{l(l+1) - (Ze^2)^2}{r^2}\,\varphi \qquad (28)$$

$$\Delta_r\varphi = \varphi'' + \frac{2}{r}\varphi \qquad (29)$$

$$\frac{\varepsilon^2 - m^2}{2m}\,\varphi = -\frac{1}{2m}\Delta_r\varphi + \left(\frac{\tilde{l}(\tilde{l}+1)}{r^2} - \frac{\tilde{Z}e^2}{r}\right)\varphi \qquad (30)$$

$$\left\{\begin{aligned} \tilde{E} &= \frac{\varepsilon^2 - m^2}{2m} \\ \tilde{l}(\tilde{l}+1) &= l(l+1) - (Ze^2)^2 \qquad (31) \\ \tilde{Z}e^2 &= Ze^2\,\frac{\varepsilon}{m} \end{aligned}\right.$$

$$\tilde{l} = \sqrt{(l+\tfrac{1}{2})^2 - Z^2e^4} - \frac{1}{2} \equiv \gamma - \frac{1}{2} \qquad (32)$$

$$\begin{cases} \tilde{\ell} = \gamma - \frac{1}{2} \\ \overset{(32)}{\gamma} = \sqrt{\left(\ell+\frac{1}{2}\right)^2 - Z^2 e^4} \end{cases} \qquad (33)$$

For the Rydberg series

the Schredinger eq gives

$$E = -\frac{m\,(Ze^2)^2}{(n_r + 1 + \ell)^2} \qquad (33)$$

$n_\ell = 0, 1 \ldots$

(30) looks similar to the Shredinger eq.

$$\Downarrow$$

Substitute in (33)

$$\begin{cases} E \to \tilde{E} \\ \ell \to \tilde{\ell} \\ Ze^2 \to \tilde{Z}e^2 \end{cases} \qquad (34)$$

(33), (34) $\Rightarrow$

$$\tilde{Z}e^2$$

$$\frac{\mathcal{E}^2 - m^2}{2m} = -\frac{m}{2}\left(\frac{\mathcal{E}}{m} Ze^2\right)^2 \cdot \frac{1}{(n_r + \tilde{\ell} + 1)^2} \qquad (35)$$

$$\tilde{E}$$

$$j - \frac{1}{2}$$

$$\varepsilon^2 - m^2 \underset{(35)}{=} \frac{\varepsilon^2 (Ze^2)^2}{\big(\underbrace{n_r + \mu + \tfrac{1}{2}}\big)^2} \qquad (36)$$

$$\underbrace{n_r + \ell + 1 + \mu - (\ell + \tfrac{1}{2})}$$

$$n + \mu - \ell - \tfrac{1}{2}$$

$$n = 1, 2, \ldots$$

$$(37)$$

$$\mathcal{E} = \frac{mc^2}{\left[1 + \frac{(Z\alpha)^2}{(n + \mu - \ell - \frac{1}{2})^2}\right]^{1/2}}$$

$$\mu = \sqrt{(\ell + \tfrac{1}{2})^2 - (Z\alpha)^2}$$

$\mathcal{E} < m$

$\mathcal{E} \in$ discrete spectrum

$$\begin{cases} n = 1, 2, 3 \ldots \\ \ell = 0, 1, 2 \ldots \end{cases}$$

Expansion in $Z\alpha$

$$\mathcal{E} = m \left[1 - \frac{(Z\alpha)^2}{2n^2} - \left(\frac{2}{2\ell+1} - \frac{3}{4n} \right) \frac{(Z\alpha)^4}{2n^3} \right] \tag{38}$$

$\mathcal{E} = mc^2$

Rydberg series ✓

Rel Corrections