Gauge

invariance

1. QED = U(1)

$$\psi \to \psi' = e^{-i\alpha}\psi \qquad (1)$$

$\alpha = \text{Const}$ — global transformation

$\alpha = \alpha(x)$ — local transformation

$$\partial_\mu \psi \to \partial_\mu \psi' = e^{-i\alpha}\left(\partial_\mu \psi - i\partial_\mu \alpha \psi\right) \qquad (2)$$

$$\begin{cases} A_\mu \to A_\mu' = A_\mu + \frac{1}{e}\partial_\mu \alpha \\ eA_\mu \to eA_\mu' = eA_\mu + \partial_\mu \alpha \end{cases} \qquad (3)$$

$$\nabla_\mu = \partial_\mu + ieA_\mu \qquad (4)$$

$$\nabla_\mu \psi \to \nabla_\mu' \psi' \underset{(4)}{=} \left(\partial_\mu + ieA_\mu'\right)\psi' =$$

$$= \left(\partial_\mu + ieA_\mu + i\partial_\mu \alpha\right)\left(e^{-i\alpha}\psi\right) = \qquad (5)$$

$$= e^{-i\alpha}\nabla_\mu \psi$$

$$\left\{\begin{array}{l} \psi \to e^{-i\alpha}\psi \\ eA_\mu \to eA_\mu + \partial_\mu \alpha \\ \nabla_\mu \psi \to e^{-i\alpha} \nabla_\mu \psi \end{array}\right. \qquad (6)$$

$$\nabla_\mu = \partial_\mu + ieA_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \qquad (7)$$

$$= \frac{-i}{e}\left[\nabla_\mu \nabla_\nu\right]$$

$$\mathcal{E} = \frac{\vec{E}^2 + \vec{B}^2}{2} \qquad (8)$$

$$\mathcal{L} = \frac{\vec{E}^2 - \vec{B}^2}{2} =$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (9)$$

Units $\quad \frac{e^2}{4\pi\hbar c} = \alpha = \frac{1}{137}$

$$(10)$$

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{E} = \frac{Q\,\vec{u}}{4\pi r^2}$$

$$A_\mu \Rightarrow \widetilde{A_\mu} = g A_\mu$$

scaling (11)

$$F_{\mu\nu} \Rightarrow \widetilde{F_{\mu\nu}} = g F_{\mu\nu}$$

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \qquad (12)$$

Non-Abelian gauge theory

SU(2)

$$\begin{cases} A_\mu^a & a = 1, 2, 3 \\ \vec{A}_\mu = \{ A_\mu^a \} \end{cases} \qquad (11)$$

$$\begin{cases} F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, \epsilon^{abc} A_\mu^b A_\nu^c \\ \vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\, \vec{A}_\mu \times \vec{A}_\nu \end{cases} \qquad (12)$$

$$\begin{cases} \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu} = \\ \quad = -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} \end{cases} \qquad (13)$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{14}$$

$$A_\mu = \frac{1}{2} \vec{\tau} \cdot \vec{A}_\mu \qquad 2 \times 2$$

$$= \frac{1}{2} \tau^a A_\mu^a \tag{15}$$

$$\psi \rightarrow \psi' = U \psi \tag{16}$$

$$\left\{ \begin{array}{l} U \in SU(2) \\ U^+ = U^{-1} \\ \det U = 1 \end{array} \right. \tag{17}$$

$$|\psi_1|^2 + |\psi_2|^2 \underset{(17)}{=} |\psi_1'|^2 + |\psi_2'|^2 \tag{18}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{14}$$

$$\psi \to \psi' = U\psi \tag{15}$$

$$\begin{cases} |\psi_1'|^2 + |\psi_2'|^2 = |\psi_1|^2 + |\psi_2|^2 \\ \psi^+\psi = \underbrace{\psi'^+\psi'}_{\psi^+ U^+ U \psi} \end{cases} \tag{16}$$

$$\begin{cases} U^+U = 1 \\ \det U = 1 \end{cases} = \quad U \in SU(2) \tag{17}$$

$$\begin{cases} U_1, U_2 \in SU(2) \\ U_1 U_2 \in SU(2) \end{cases} \tag{18}$$

$$U \in SU(2) \Rightarrow U^{-1} \in SU(2) \tag{19}$$

$$U = U(x) \tag{20}$$

$$A_\mu = A_\mu^a \left(\frac{\tau^a}{2}\right) \tag{21}$$

$$\nabla_\mu = \partial_\mu - i g A_\mu \tag{22}$$

definition

$$\psi \to \psi' = U\psi \tag{23}$$

we want

$$\nabla_\mu \psi \to \nabla_\mu' \psi' = U \nabla_\mu \psi \tag{24}$$

$$\nabla_\mu' = \partial_\mu - i g A_\mu' \tag{25}$$

$$A_\mu' = ?$$

See (28)

find A_μ' which satisfies (24)

$$\left(\partial_\mu - i\underset{\smile}{g} A'_\mu\right) U \psi \underset{(23,25)}{=} U \left(\partial_\mu - i\underset{\smile}{g} A_\mu\right) \psi \qquad (26)$$

$$\Downarrow$$

$$U \partial_\mu \psi + \partial_\mu U \psi - i\underset{\smile}{g} A'_\mu U \psi =$$

$$= U \left[\partial_\mu \psi + U^{-1} \partial_\mu U \psi - i\underset{\smile}{g}\, U^{-1} A'_\mu U \psi \right]$$

$$= U \left[\partial_\mu - i\underset{\smile}{g} \underbrace{\left(U^{-1} A'_\mu U + \frac{i}{\underset{\smile}{g}} U^{-1} \partial_\mu U \right)}_{A_\mu} \right] \psi$$

$$\Downarrow$$

$$U^{-1} A'_\mu U = A_\mu - \frac{i}{\underset{\smile}{g}} U^{-1} \partial_\mu U \qquad (27)$$

$$A'_\mu = U A_\mu U^{-1} - \underbrace{\frac{i}{g} \partial_\mu U U^{-1}}_{\frac{i}{g} U \partial_\mu U^{-1}} \qquad (28)$$

$$\boxed{g A'_\mu = U g A_\mu U^{-1} + i U \partial_\mu U^{-1}} \qquad (29)$$

Compare QED

Take $\underset{\text{Eq. (1)}}{U = e^{-i\alpha}}$ (30)

$$i U \partial_\mu U^{-1} = - \partial_\mu \alpha$$

see (3)

Eq. (30) shows that (29) is in line with (3)

Sign is different because definitions (25) and (4) for YM and QED ~~one~~ differ in sign

Theorem

If
$$A_\mu^+ = A_\mu \qquad (31)$$
$$\mathrm{Tr}\, A_\mu = 0$$

Then
$$\left\{ \begin{aligned} A_\mu'^{+} &= A_\mu' \\ \mathrm{Tr}\, A_\mu' &= 0 \end{aligned} \right. \qquad (32)$$

$$\left\{ \begin{aligned} A_\mu &= A_\mu^a \frac{\tau^a}{2} \\ A_\mu' &= A_\mu'^a \frac{\tau^a}{2} \end{aligned} \right. \qquad (33)$$

(32)

$$(32, 33) \Rightarrow \quad A_\mu'^a \text{ is a real vector} \qquad (34)$$

Definition

If $\begin{cases} A = 2\times 2 \text{ matrix} \\ A^{+} = A \\ \mathrm{Tr}\, A = 0 \end{cases}$ (31)

Then $A \in su(2)$ algebra (32)

Compare (17, 18, 19) for $SU(2)$ group.

(32) $\Rightarrow$ $\begin{cases} \text{If } A, B \in su(2) \\ \text{Then } \alpha A + \beta B \in su(2) \\ \text{where } \alpha, \beta \text{ real numbers} \end{cases}$ (33)

$$\begin{cases} \psi' = U\psi \\ \nabla'_\mu \psi' = U \nabla_\mu \psi \end{cases} \qquad (34)$$

...

$$\nabla'_\mu \psi' \underset{(24)}{=} U \nabla_\mu U^{-1} \underbrace{U\psi}_{\psi'} \qquad (35)$$

$$\nabla'_\mu \underset{(35)}{=} U \nabla_\mu U^{-1} \qquad (36)$$

definition (compare (7))

$$F_{\mu\nu} = \frac{+i}{g} \left[\nabla_\mu \nabla_\nu\right] =$$

$$= \frac{i}{g} \left[\partial_\mu - igA_\mu ,\ \partial_\nu - igA_\nu\right]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \qquad (37)$$

$$\boxed{F_{\mu\nu} \underset{(37)}{=} \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu A_\nu]} \qquad (38)$$

$$\nabla_\mu^{+} \underset{(40,41,42)}{=} -\nabla_\mu \qquad (39)$$

$$\partial_\mu^{+} = -\partial_\mu \qquad (40)$$

$$A_\mu^{+} = A_\mu \qquad (41)$$

$$\nabla_\mu = \partial_\mu - ig A_\mu \qquad (42)$$

$$F_{\mu\nu}^{+} = F_{\mu\nu} \qquad (43)$$

(43) follows directly from def.

$$F_{\mu\nu} = \frac{i}{g}[\nabla_\mu \nabla_\nu]$$

$$F_{\mu\nu} \underset{(43)}{=} F_{\mu\nu}^{a} \frac{\tau^a}{2} \qquad (44)$$

$$\boxed{F_{\mu\nu}^{a} \underset{(38,44)}{=} \partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a} + g\,\epsilon^{abc} A_\mu^{b} A_\nu^{c} \qquad (45)}$$

$$F_{\mu\nu}' = \frac{i}{g}\left[\nabla_\mu' \, \nabla_\nu'\right] =$$

$$= \frac{i}{g}\left[U \nabla_\mu U^{-1}, \; U \nabla_\nu U^{-1}\right]$$

$$= \frac{i}{g} \; U \left[\nabla_\mu \, \nabla_\nu\right] U^{-1}$$

$$= U F_{\mu\nu} U^{-1} \qquad (46)$$

definition

$$\mathcal{L} = -\frac{1}{4k}\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$$k = 2$$

$$= -\frac{1}{4}\,F_{\mu\nu}^{a}\,F^{a\,\mu\nu} \qquad (47)$$

$$\mathcal{L} \to \mathcal{L}' \underset{(49)}{=} \mathcal{L} \qquad (48)$$

$$\mathcal{L}' = -\frac{1}{4k}\,\mathrm{Tr}\left(F'_{\mu\nu}F'^{\mu\nu}\right)$$

$$= -\frac{1}{4k}\,\mathrm{Tr}\left(U F_{\mu\nu} \underbrace{U^{-1} U}_{1} F^{\mu\nu} U^{-1}\right)$$

$$= -\frac{1}{4k}\,\mathrm{Tr}\left(U F_{\mu\nu} F^{\mu\nu} U^{-1}\right) = \mathcal{L} \qquad (49)$$

Summary for SU(2)

$$\psi \to \psi' \underset{(23)}{=} U\psi$$

$$\nabla_\mu \psi \to \nabla'_\mu \psi' \underset{(24)}{=} U \nabla_\mu \psi$$

$$F_{\mu\nu} \to F'_{\mu\nu} \underset{(46)}{=} U F_{\mu\nu} U^{-1} \quad (50)$$

$$\mathcal{L} \to \mathcal{L}' \underset{(4\cancel{8})}{\equiv} \mathcal{L}$$

$$\psi^+ \nabla_\mu \psi \to \psi'^+ \nabla'_\mu \psi' \equiv \psi^+ \nabla_\mu \psi$$

SU(n)

$$A_\mu = n \times n \qquad (51)$$

$$\begin{cases} A_\mu^+ = A_\mu \\ \mathrm{Tr}\, A_\mu = 0 \end{cases} \qquad A_\mu \in su(n) \qquad (52)$$

$$\psi = n\text{-spinor} \qquad (53)$$

$$\nabla_\mu = \partial_\mu - i g A_\mu \qquad (54)$$

$$F_{\mu\nu} = \frac{i}{g}\left[\nabla_\mu \nabla_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu + g\left[A_\mu , A_\nu\right] \qquad (55)$$

$$\mathcal{L} = -\frac{1}{4}\,\frac{1}{k}\,\mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \qquad (56)$$

$$\begin{cases} U = n \times n \\ U^+ = U^{-1} \\ \det U = 1 \end{cases} \Rightarrow U \in SU(n) \quad (57)$$

$$\begin{cases} A_\mu = n \times n \\ A_\mu^+ = A_\mu \\ \mathrm{Tr}\, A_\mu = 0 \end{cases} \Rightarrow A_\mu \in su(n)$$

$$\begin{cases} F_{\mu\nu} = n \times n \\ F_{\mu\nu}^+ = F_{\mu\nu} \\ \mathrm{Tr}\, F_{\mu\nu} = 0 \end{cases} \quad F_{\mu\nu} \in su(n)$$

$$
\boxed{
\begin{aligned}
\psi &\to U\psi \\
\nabla_\mu \psi &\to U \nabla_\mu \psi \\
\nabla_\mu &\to U \nabla_\mu U^{-1} \\
F_{\mu\nu} &\to U F_{\mu\nu} U^{-1} \\
\mathcal{L} &\to \mathcal{L}
\end{aligned}
}
\tag{57}
$$

Number of gauge bosons

$$\begin{cases} A_\mu = A_\mu^+ \\ \mathrm{Tr}\, A_\mu = 0 \end{cases} \tag{58}$$

fix μ think about colour degrees

$$\begin{cases} \text{Off diagonal} & 2 \cdot \frac{n(n-1)}{2} = n(n-1) \\ \text{Diagonal} & n-1 \end{cases} \tag{59}$$

$$\text{Total} \quad N \underset{(59)}{=} n-1 + n(n-1) = \underline{n^2 - 1} \tag{60}$$

For $SU(2)$, $n=2$: $\quad \underline{N = 3} \quad W^{\pm}, Z$ (61)

For $SU(3)$, $n=3$: $\quad \underline{N = 8} \quad$ 8 colours for gluons (62)