Vacuum polarization
Uehling potential
Landau pole

$$\Delta\varphi(r) = -4\pi\rho(r) \tag{1}$$

$$\rho(\mathbf{r}) = e\sum_{\sigma=1,2}\int\frac{d^3k}{(2\pi)^3}\,\psi_{\mathbf{k}}^{*}(\mathbf{r})\,\psi_{\mathbf{k}}(\mathbf{r}),\qquad e=-|e|<0 \tag{2}$$

Here the summation runs over all the states in the lower continuum ε < 0. If we consider a potential U(r) as a perturbation then Eq.(2) yeilds

$$\Delta\,\delta\varphi(r) = -4\pi\,\delta\rho(r) \tag{3}$$

$$\delta\rho(\mathbf{r}) = 2e\,\mathrm{Re}\sum_{\sigma=1,2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\psi_{\mathbf{k}}^{*}(\mathbf{r})\,\delta\psi_{\mathbf{k}}(\mathbf{r}) \tag{4}$$

$$\gamma_\mu\,(p^\mu - eA^\mu)\,\psi = m\,\psi \tag{5}$$

$$\boldsymbol{\gamma_\mu\,\gamma_\nu\,(p^\mu - eA^\mu)\,(p^\nu - eA^\nu)\,\psi = m^2\,\psi}$$

$$\boldsymbol{\gamma_\mu\,\gamma_\nu = g_{\mu\nu} + \sigma_{\mu\nu}}$$

$$\gamma_\mu \gamma_\nu (p^\mu - eA^\mu)(p^\nu - eA^\nu)\psi = (p - eA)^2\psi + \sigma_{\mu\nu}(p^\mu - eA^\mu)(p^\nu - eA^\nu)\psi = (p - eA)^2\psi + \frac{1}{2}\sigma_{\mu\nu}[p^\mu - eA^\mu, p^\nu - eA^\nu]\psi = (p - eA)^2\psi - \frac{i}{2} e\,\sigma_{\mu\nu} F^{\mu\nu}\psi = m^2\psi$$

$$(p - eA)^2\psi - \frac{i}{2} e\,\sigma_{\mu\nu} F^{\mu\nu}\psi = m^2\psi \qquad (6)$$

$$((p - eA)^2 - m^2)\psi + e(\mathbf{\Sigma}\cdot\mathbf{B} - i\,\boldsymbol{\alpha}\cdot\mathbf{E})\psi = 0 \qquad (6)$$

$$U = -\frac{Z e^2}{r}, \quad \mathbf{F} = -\frac{Z e^2 \mathbf{r}}{r^3} \qquad (7)$$

Static electric field: $i\frac{\partial}{\partial t} - eA_0 \to \varepsilon - U$, $e\,\mathbf{E} \to \mathbf{F}$

$$((\varepsilon - U)^2 - m^2 + \Delta - i\,\boldsymbol{\alpha}\cdot\mathbf{F})\psi = 0, \qquad (8)$$

Lower continuum $\to$ $\varepsilon = -\varepsilon_{\mathbf{k}}$

$$((\varepsilon_{\mathbf{k}} + U)^2 - m^2 + \Delta - i\,\boldsymbol{\alpha}\cdot\mathbf{F})\psi_{\mathbf{k}} = 0, \qquad (9)$$

$$(\varepsilon_{\mathbf{k}}^2 - m^2 + \Delta)\,\delta\psi_{\mathbf{k}}(\mathbf{r}) = (-2\,\varepsilon_{\mathbf{k}} U + i\,\boldsymbol{\alpha}\cdot\mathbf{F})\psi_{\mathbf{k}}(\mathbf{r}) \qquad (10)$$

The term with U arises due to the electron charge. The sign minus in $-2\varepsilon_{\mathbf{k}}U$ modifies the influence of the potential. If the potential produces attraction U<0, then for the states in the lower continuum is represents *repulsion*.
The term with F arises from the contribution of the spin. We will see that this contribution results in the effective *attraction*.

$$\psi_{\mathbf{k}}(\mathbf{r}) = u_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{r}}, \qquad u_{\mathbf{k}}^{*}\, u_{\mathbf{k}} = 1. \tag{11}$$

The normalization is conventional for usual quantum mechanical

problems; it is different from what is often assumed in QED

$\bar{u}_{\mathbf{k}}\, u_{\mathbf{k}} = \pm 2\, m.$

$$(\mathbf{k}^2 + \Delta)\, \delta\psi_{\mathbf{k}}(\mathbf{r}) = (-2\, \varepsilon_{\mathbf{k}}\, U + i\, \boldsymbol{\alpha}\cdot\mathbf{F})\, u_{\mathbf{k}}\, e^{i\,\mathbf{k}\cdot\mathbf{r}} \tag{12}$$

$$\delta\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\mathbf{k}^2 + \Delta + i0}\left[e^{i\,\mathbf{k}\cdot\mathbf{r}}\,(-2\, \varepsilon_{\mathbf{k}}\, U + i\, \boldsymbol{\alpha}\cdot\mathbf{F})\right] u_{\mathbf{k}} \tag{13}$$

$$\delta\rho(\mathbf{r}) = 2\, e\, \mathrm{Re} \sum_{\sigma=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\,\mathbf{k}\cdot\mathbf{r}} \frac{1}{\mathbf{k}^2 + \Delta + i0}\left[e^{i\,\mathbf{k}\cdot\mathbf{r}}\,(-2\, \varepsilon_{\mathbf{k}}\, U\,(u_{\mathbf{k}}^{*}\, u_{\mathbf{k}}) + i\,(u_{\mathbf{k}}^{*}\, \boldsymbol{\alpha}\, u_{\mathbf{k}})\cdot\mathbf{F})\right] \tag{14}$$

$$u^{*} u = 1 \;\Rightarrow\; \bar{u}\, \gamma_0\, u = 1 \;\Rightarrow\; \bar{u}\, \gamma^{\mu}\, u = \frac{p^{\mu}}{\varepsilon}, \quad \varepsilon = -\varepsilon_{\mathbf{k}} \tag{15}$$

$$u^{*} u = \bar{u}_{\mathbf{k}}\, \gamma_0\, u_{\mathbf{k}} = 1, \qquad u_{\mathbf{k}}^{*}\, \boldsymbol{\alpha}\, u_{\mathbf{k}} = \bar{u}_{\mathbf{k}}\, \boldsymbol{\gamma}\, u_{\mathbf{k}} = -\frac{\mathbf{k}}{\varepsilon_{\mathbf{k}}} \tag{15}$$

$$\delta\rho(\mathbf{r}) = 2\,e\,\mathrm{Re}\sum_{\sigma=1,2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,\frac{1}{\mathbf{k}^2+\Delta+i0}\left[e^{i\mathbf{k}\cdot\mathbf{r}}\left(-2\,\varepsilon_{\mathbf{k}}\,U - i\,\frac{\mathbf{k}\cdot\mathbf{F}}{\varepsilon_{\mathbf{k}}}\right)\right] \tag{16}$$

$$\mathbf{F} = -\nabla U \tag{16}$$

$$\delta\rho(\mathbf{r}) = -4\,e\,\mathrm{Re}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,\frac{1}{\mathbf{k}^2+\Delta+i0}\left[e^{i\mathbf{k}\cdot\mathbf{r}}\left(\frac{2\,\varepsilon_{\mathbf{k}}^2 - i\,\mathbf{k}\cdot\nabla}{\varepsilon_{\mathbf{k}}}\right)U\right] \tag{16}$$

$$\text{Here}\ \sum_{\sigma=1,2}\to 2 \tag{17}$$

$$U(\mathbf{r}) = e\int U(\mathbf{q})\,e^{i\mathbf{q}\cdot\mathbf{r}}\,\frac{d^3\mathbf{q}}{(2\pi)^3}\,, \tag{18}$$

$$U(\mathbf{r}) = U^*(\mathbf{r}) \tag{19}$$

$$U(\mathbf{q}) = U(-\mathbf{q}) = U(q) \tag{19}$$

$$U = -\frac{Ze^2}{r}\,,\quad U(\mathbf{q}) = -\frac{4\pi Z e^2}{\mathbf{q}^2} \tag{20}$$

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, \frac{1}{\mathbf{k}^2+\Delta+i0}\left[\, e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\, \frac{2\,\varepsilon_{\mathbf{k}}^2+\mathbf{k}\cdot\mathbf{q}}{\varepsilon_{\mathbf{k}}}\,\right] = e^{i\mathbf{q}\cdot\mathbf{r}} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{\mathbf{k}^2-(\mathbf{k}+\mathbf{q})^2+i0}\, \frac{2\,(\mathbf{k}^2+m^2)+\mathbf{k}\cdot\mathbf{q}}{\sqrt{\mathbf{k}^2+m^2}} \tag{21}$$

Definition (22)

$$-\frac{P(\mathbf{q}^2)}{4\pi} = -4\, e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{\mathbf{k}^2-(\mathbf{k}+\mathbf{q})^2+i0}\, \frac{2\,(\mathbf{k}^2+m^2)+\mathbf{k}\cdot\mathbf{q}}{\sqrt{\mathbf{k}^2+m^2}} \tag{22}$$

$$P(\mathbf{q}^2) = P(q^2) \tag{23}$$

$$e\,\delta\rho(\mathbf{r}) = \mathrm{Re}\int \left(-\frac{P(q)}{4\pi}\right) U(q)\, e^{i\mathbf{q}\cdot\mathbf{r}}\, \frac{d^3\mathbf{q}}{(2\pi)^3} \tag{24}$$

$$e\,\delta\varphi(r) \equiv \delta U = \mathrm{Re}\int \left(-\frac{P(q^2)}{4\pi}\right) \frac{4\pi}{\mathbf{q}^2}\, U(q)\, e^{i\mathbf{q}\cdot\mathbf{r}}\, \frac{d^3\mathbf{q}}{(2\pi)^3} = -\mathrm{Re}\int \frac{P(q^2)}{\mathbf{q}^2}\, U(q)\, e^{i\mathbf{q}\cdot\mathbf{r}}\, \frac{d^3\mathbf{q}}{(2\pi)^3} \tag{25}$$

$$\delta U = 4\pi Z\, e^2\, \mathrm{Re}\int \frac{P(q^2)}{(\mathbf{q}^2)^2}\, e^{i\mathbf{q}\cdot\mathbf{r}}\, \frac{d^3\mathbf{q}}{(2\pi)^3} \tag{26}$$

$$-\frac{P(q^2)}{4\pi} = -4\ e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{1}{\mathbf{k}^2 - (\mathbf{k}+\mathbf{q})^2 + i0}\ \frac{2(\mathbf{k}^2+m^2) + \mathbf{k}\cdot\mathbf{q}}{\sqrt{\mathbf{k}^2+m^2}} =$$
$$4\ e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{1}{\mathbf{q}^2 + 2\,\mathbf{q}\cdot\mathbf{k} - i0}\ \frac{2(\mathbf{k}^2+m^2) + \mathbf{k}\cdot\mathbf{q}}{\sqrt{\mathbf{k}^2+m^2}} =$$
$$4\ e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\frac{2(\mathbf{k}^2+m^2) - \mathbf{q}^2/2}{\mathbf{q}^2 + 2\,\mathbf{q}\cdot\mathbf{k} - i0} + \frac{1}{2}\right) \frac{1}{\sqrt{\mathbf{k}^2+m^2}} \quad (27)$$

$$-\frac{P(q^2)}{4\pi} = 8\ e^2 \int \frac{d^3k}{(2\pi)^3} \left(\frac{k^2+m^2-q^2/4}{q^2 + 2\,\mathbf{q}\cdot\mathbf{k} - i0}\right) \frac{1}{\sqrt{k^2+m^2}} + \text{const},$$
$$\text{const} = 2\ e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{1}{\sqrt{\mathbf{k}^2+m^2}} \quad (28)$$

Renormalization (see also below) yeilds

$$\text{const} \Rightarrow 0 \quad (29)$$

$$-\frac{P(q^2)}{4\pi} = 8\ e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{k^2+m^2-q^2/4}{q^2 + 2\,\mathbf{q}\cdot\mathbf{k} - i0}\ \frac{1}{\sqrt{k^2+m^2}} \quad (30)$$

$$-\frac{P(q^2)}{4\pi} = \frac{8\cdot 2\pi}{(2\pi)^3} e^2 \int_0^\infty dk \int_{-1}^{1} dz \frac{k^2+m^2-q^2/4}{q^2+2qkz-i0} \frac{k^2}{\sqrt{k^2+m^2}} = \tag{31}$$

$$\frac{e^2}{\pi^2 q} \int_{-\infty}^{\infty} dk \int_{-1}^{1} dz \frac{k^2+m^2-q^2/4}{q+2kz-i0} \frac{k^2}{\sqrt{k^2+m^2}} =$$

$$\frac{e^2}{\pi^2 q} \int_{-\infty}^{\infty} dk$$

$$\frac{1}{2k} \ln\left(\frac{q+2k-i0}{q-2k-i0}\right)(k^2+m^2-q^2/4) \frac{k^2}{\sqrt{k^2+m^2}} \tag{31}$$

$$= \frac{e^2}{2\pi^2} \int_{-\infty}^{\infty} \frac{1}{q} \ln\left(\frac{q/2+k-i0}{q/2-k-i0}\right)(k^2+m^2-q^2/4) \frac{k\,dk}{\sqrt{k^2+m^2}} \tag{31}$$

$$-P(q^2) = \frac{2e^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{q} \ln\left(\frac{q/2+k-i0}{q/2-k-i0}\right)(k^2+m^2-q^2/4) \frac{k\,dk}{\sqrt{k^2+m^2}} \tag{32}$$

$$-P\left(q^2\right)=\frac{2e^2}{\pi}\int_{-\infty}^{\infty}\frac{1}{q}\ln\left(\frac{q/2+k-i0}{q/2-k-i0}\right)d\left[\left(\frac{k^2+m^2}{3}-\frac{q^2}{4}\right)\sqrt{k^2+m^2}\right]=$$
$$-\frac{2e^2}{\pi}\int_{-\infty}^{\infty}\frac{1}{q}\left(\frac{1}{q/2+k-i0}+\frac{1}{q/2-k-i0}\right) \tag{33}$$

$$\left[\left(\frac{k^2+m^2}{3}-\frac{q^2}{4}\right)\sqrt{k^2+m^2}\right]dk=$$
$$-\frac{2e^2}{\pi}\int_{-\infty}^{\infty}\frac{1}{q^2/4-k^2-i0}\left(\frac{k^2+m^2}{3}-\frac{q^2}{4}\right)\sqrt{k^2+m^2}\,dk \tag{33}$$

Again throwing away a const (having in mind the renormalization, see below)

$$P\left(q^2\right)=\frac{2e^2}{\pi}\int_{-\infty}^{\infty}\frac{1}{q^2/4-k^2-i0}\left(\frac{m^2-2k^2}{3}\right)\sqrt{k^2+m^2}\,dk=$$
$$\frac{2e^2}{3\pi}\int_{-\infty}^{\infty}\frac{1}{q^2/4-k^2-i0}\left(m^2-2k^2\right)\sqrt{k^2+m^2}\,dk \tag{34}$$

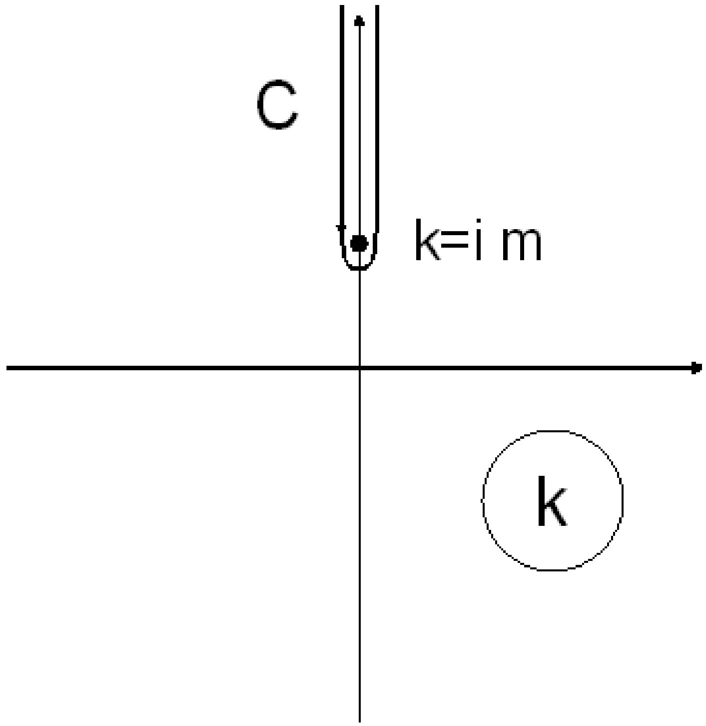

Figure 1

Shift the integration counter C from the real axes into the upper semiplane of the complex plane k. The point k = i m on the imaginary axes is a branch point. Make the integration counter C to begin at k=i∞, going down to k=i m, then circle clockwise and return back to k=i∞. Making a substitution k→i k one finds

$$P(q^2) = i^2 \frac{4}{3\pi} e^2 \int_m^\infty \frac{1}{q^2/4 + k^2} (m^2 + 2k^2) \sqrt{k^2 - m^2}\, dk \tag{35}$$

Here an additional coefficient 2 comes from the two branches of the counter.

$$P(q^2) = -\frac{16\,e^2}{3\,\pi}\int_m^\infty \frac{1}{q^2+4\,k^2}\,(m^2+2\,k^2)\,\sqrt{k^2-m^2}\;dk \tag{36}$$

$$4\,k^2 = t' \tag{37}$$

$$k = \frac{1}{2}\sqrt{t'} \tag{37}$$

$$dk = \frac{1}{4}\,\frac{dt'}{\sqrt{t'}} \tag{37}$$

$$P(q^2) \to -\frac{16}{3\,\pi}\,e^2\,\frac{1}{4}\,\frac{1}{2}\,\frac{1}{2}\int_{4\,m^2}^\infty \frac{1}{q^2+t'}\,(2\,m^2+t')\,\sqrt{t'-4\,m^2}\;\frac{dt'}{\sqrt{t'}} =$$

$$-\frac{e^2}{3\,\pi}\int_{4\,m^2}^\infty \frac{1}{q^2+t'}\,(2\,m^2+t')\,\sqrt{t-4\,m^2}\;\frac{dt'}{\sqrt{t'}} = \tag{38}$$

$$P(q^2) = -\frac{e^2}{3\,\pi}\int_{4\,m^2}^\infty \frac{1}{q^2+t'}\,(t'+2\,m^2)\,\sqrt{\frac{t'-4\,m^2}{t'}}\;dt'$$

$$P_{Reg}\,(q^2) = P\,(q^2) - P\,(0) - q^2\,P'\,(0) \tag{39}$$

The constant was thrown away a couple of times above. This makes no

difference, since the renormalization takes the constant away anyway.

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Simplify[1/(q^2 + t') - Normal[Series[1/(q^2 + t'), {q, 0, 2}]]]
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$$\frac{q^4}{(t')^2\,(q^2+t')}$$

Thus, regularization means (40)

$$\frac{1}{q^2+t'} \to \frac{q^4}{(t')^2\,(q^2+t')} \tag{40}$$

$$P_{Reg}\,(q^2) = -\frac{e^2}{3\pi}\int_{4m^2}^{\infty}\frac{q^4}{(t')^2\,(q^2+t')}\,(t'+2m^2)\,\sqrt{\frac{t'-4m^2}{t'}}\,dt' \tag{41}$$

$$P\,(q^2) = -\frac{\alpha}{3\pi}\,q^4\int_{4m^2}^{\infty}\frac{1}{t'+q^2-i0}\,\frac{t'+2m^2}{t'^2}\,\sqrt{\frac{t'-4m^2}{t'}}\,dt' \tag{42}$$

$$t = -\mathbf{q}^2 \tag{43}$$

$$\mathcal{P}\,(t) = \mathcal{P}\,(-\mathbf{q}^2) = P\,(\mathbf{q}^2) \tag{43}$$

$$t = -\mathbf{q}^2,$$

$$\mathcal{P}(t) = -\frac{\alpha}{3\pi} t^2 \int_{4m^2}^{\infty} \frac{1}{t' - t - i0} \frac{t' + 2m^2}{t'^2} \sqrt{\frac{t' - 4m^2}{t'}} \, dt' \tag{44}$$

$$t' = 4m^2 \zeta^2 \tag{45}$$

$$P(q^2) = -\frac{e^2}{3\pi} q^4 \int_1^{\infty} \frac{1}{4m^2\zeta^2 + q^2} \frac{\zeta^2 + 1/2}{\zeta^4} \frac{\sqrt{\zeta^2 - 1}}{\zeta} 2\zeta \, d\zeta = -\frac{2e^2}{3\pi} q^4 \int_1^{\infty} \frac{1}{4m^2\zeta^2 + q^2} \frac{\zeta^2 + 1/2}{\zeta^4} \sqrt{\zeta^2 - 1} \, d\zeta \tag{45}$$

$$P(q^2) = -\frac{2\alpha}{3\pi} q^4 \int_1^{\infty} \frac{1}{4m^2\zeta^2 + q^2} \left(1 + \frac{1}{2\zeta^2}\right) \frac{\sqrt{\zeta^2 - 1} \, d\zeta}{\zeta^2} \tag{46}$$

$$P(q^2) \simeq -\frac{\alpha}{3\pi} q^2 \ln\left(\frac{q^2}{m^2}\right), \quad q^2 \gg m^2 \tag{47}$$

For spinor particles it was found that

$$P\left(q^2\right) = -\frac{\alpha}{3\pi} q^4 \int_{4m^2}^{\infty} \frac{1}{t' + q^2 - i0} \frac{t' + 2m^2}{t'^2} \sqrt{\frac{t' - 4m^2}{t'}} \, dt' = \\ -\frac{\alpha}{3\pi} q^4 \int_{4m^2}^{\infty} \frac{1}{t' + q^2 - i0} \frac{2m^2 - t'/2 + 3t'/2}{t'^2} \sqrt{\frac{t' - 4m^2}{t'}} \, dt' \tag{48}$$

Here m^2-t'/2 comes from the charge contribution, while 3t'/2 originates from the spin. These two terms have different signs. The charge tends to make P(q) positive, while the spin results in a opposite, negative contribution, which dominates.

Calculating the polarization for scalar particles one needs to make three amendments:
1. To change the total sign, which is related to statistics (i.e. to eliminate the negative energy)
2. To eliminate the spin contribution, i.e. the term 3t'/2.
3.To reduce the result by a factor of 2, which exists for fermions due to their spin 1/2, but is absent for scalars.
Then the polarization P_0 created by scalar particles reads

$$P_0\left(q^2\right) = -\left(-\frac{1}{2}\right) \frac{\alpha}{3\pi} q^4 \int_{4m^2}^{\infty} \frac{1}{t' + q^2 - i0} \frac{2m^2 - t'/2}{t'^2} \sqrt{\frac{t' - 4m^2}{t'}} \, dt' = \\ -\frac{1}{2} \frac{\alpha}{6\pi} q^4 \int_{4m^2}^{\infty} \frac{1}{t' + q^2 - i0} \frac{t' - 4m^2}{t'^2} \sqrt{\frac{t' - 4m^2}{t'}} \, dt' \tag{49}$$

$$P_0\left(q^2\right) \simeq -\frac{\alpha}{12\pi} q^2 \ln\left(\frac{q^2}{m^2}\right), \quad q^2 = \mathbf{q}^2 \gg m^2 \tag{50}$$

The sign of the vacuum polarization is same for fermions (Landau, Abrikosov, Khalatnikov, 1954)) and scalars (Fradkin, 1955?).

$$\delta U\ (r) = -4\pi Z e^2\ \mathrm{Re}\int \frac{P\ (q^2)}{(\mathbf{q}^2)^2}\ e^{i\mathbf{q}\cdot\mathbf{r}}\ \frac{d^3\mathbf{q}}{(2\pi)^3} \qquad (51)$$

$$P\ (q^2) = -\frac{2\alpha}{3\pi}\ q^4 \int_1^{\infty} \frac{1}{4\ m^2\ \zeta^2 + q^2}\ \left(1 + \frac{1}{2\ \zeta^2}\right)\ \frac{\sqrt{\zeta^2 - 1}}{\zeta^2}\ d\zeta \qquad (51)$$

$$\int \frac{1}{q^2 + \mu^2}\ e^{i\mathbf{q}\cdot\mathbf{r}}\ \frac{d^3\mathbf{q}}{(2\pi)^3} = \frac{\exp\ (-\mu r)}{4\pi r} \qquad (52)$$

$$\delta U = 4\pi Z e^2\ \mathrm{Re}\int \frac{P\ (q^2)}{(\mathbf{q}^2)^2}\ e^{i\mathbf{q}\cdot\mathbf{r}}\ \frac{d^3\mathbf{q}}{(2\pi)^3} \qquad (53)$$

$$\delta U\ (r) = 4\pi Z e^2 \left(-\frac{2\alpha}{3\pi}\right) \int_1^{\infty} d\zeta \int \frac{d^3\mathbf{q}}{(2\pi)^3}\ \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{4\ m^2\ \zeta^2 + q^2}\ \left(1 + \frac{1}{2\ \zeta^2}\right)\ \frac{\sqrt{\zeta^2 - 1}}{\zeta^2} = -\frac{Z e^2}{r}\ \frac{2\alpha}{3\pi} \int_1^{\infty} d\zeta\ \left(1 + \frac{1}{2\ \zeta^2}\right)\ \frac{\sqrt{\zeta^2 - 1}}{\zeta^2}\ e^{-2\,m r \zeta} \qquad (54)$$

$$U\ (r) = -\frac{Z\ e^2}{r} \qquad (55)$$

$$U\ (r) + \delta U = -\frac{Z e^2}{r}\left(1 + \frac{2\alpha}{3\pi}\int_1^{\infty}\left(1 + \frac{1}{2\ \zeta^2}\right)\ \frac{\sqrt{\zeta^2 - 1}}{\zeta^2}\ e^{-2\,m r \zeta}\ d\zeta\right) \qquad (55)$$

Consider $r \gg \frac{1}{m}$ (56)

$$\delta U \simeq -\frac{Ze^2}{r}\frac{2\alpha}{3\pi}\int_1^\infty d\zeta\left(1+\frac{1}{2}\right)\sqrt{2(\zeta-1)}\ e^{-2mr\zeta} =$$

$$-\frac{Ze^2}{r}\frac{\sqrt{2}\,\alpha}{\pi}\int_1^\infty d\zeta\sqrt{\zeta-1}\ e^{-2mr\zeta} \underset{\zeta\to\zeta+1}{=}$$

$$-\frac{Ze^2}{r}\frac{\sqrt{2}\,\alpha}{\pi}\int_0^\infty d\zeta\sqrt{\zeta}\ e^{-2mr(\zeta+1)} \underset{\zeta=x^2}{=} \tag{56}$$

$$-\frac{Ze^2}{r}\frac{\sqrt{2}\,\alpha}{\pi}e^{-2mr}\,2\int_0^\infty dx\,x^2\ e^{-2mrx^2} =$$

$$-\frac{Ze^2}{r}\frac{\sqrt{2}\,\alpha}{\pi}e^{-2mr}\left(-\frac{d}{d\lambda}\right)\sqrt{\frac{\pi}{\lambda}}\ \Bigg|_{\lambda=2mr} =$$

$$-\frac{Ze^2}{r}\frac{\alpha}{\sqrt{2\pi}}\frac{e^{-2mr}}{(2mr)^{3/2}}$$

$$\delta U \simeq -\frac{Ze^2}{r}\frac{\alpha}{\sqrt{2\pi}}\frac{e^{-2mr}}{(2mr)^{3/2}},\quad r \gg \frac{1}{m} \tag{57}$$

$$\delta U \simeq -\frac{Z e^2}{r} \frac{2\alpha}{3\pi} \ln\left(\frac{1}{mr}\right), \qquad r \ll \frac{1}{m} \tag{58}$$

$$e^2(r) \simeq e^2\left(1 + \frac{2\alpha}{3\pi} \ln\left(1 + \frac{1}{mr}\right)\right) \tag{59}$$

More accurate account of polarization can be obtained by iterating the vacuum polarization, which gives

$$e^2(r) \simeq \frac{e^2}{1 - \frac{2\alpha}{3\pi} \ln\left(1 + \frac{1}{mr}\right)} \tag{60}$$

In the momentum representation (for spinors)

$$e^2(q) \simeq \frac{e^2}{1 - \frac{\alpha}{3\pi} \ln\left(q^2/m^2\right)}, \quad |q^2| \gg m^2 \tag{61}$$

For scalars

$$e^2(q) \simeq \frac{e^2}{1 - \frac{\alpha}{12\pi} \ln\left(q^2/m^2\right)}, \quad |q^2| \gg m^2 \tag{62}$$
