Schwinger pair production

Introduction

Consider the homogeneous static electric field E, which produces the force F= (-e)E on the electron, which is positive along the *x*-direction

$$\vec{F} = (F, 0, 0), \quad F > 0. \tag{1}$$

The electron charge is called here (-e), presuming that e>0. Describe this force by the potential energy U

$$U = -F x \tag{2}$$

Electron energy and momentum

$$(\varepsilon - U)^2 - p^2 = m^2 \tag{3}$$

$$p = p(x) = \left((\varepsilon + F x)^2 - m^2 \right)^{1/2} \tag{4}$$

Classical motion is allowed in the region

$$(\varepsilon + F x)^2 \geq m^2 \tag{5}$$

which splits in two, either

$$\varepsilon + F x \leq -m, \quad -\infty < x \leq \frac{-m - \varepsilon}{F} \tag{6}$$

or altenatively

$$\varepsilon + F x \geq m, \quad \frac{m - \varepsilon}{F} \leq x < \infty \tag{7}$$

In the first region, when $\varepsilon + e\,E\,x \leq -m$, the electron lives in the Dirac see, i.e. it is in the vacuum. To see this more clearly think of the limit $E \to 0$, in which $\varepsilon \leq -m$.

In the second region, where $\varepsilon + e\,E\,x \geq m$, the electron occupies the conventional electron state in the upper continuum, compare $\varepsilon \geq m$ for $E \to 0$.

The intermediate region

$$-m < \varepsilon + e\,E\,x < m \tag{8}$$

is classically forbidden

A transition of the electron from the Dirac see (first region) into the conventional upper continuum (second region) describes the pair creation. In order to fulfill this transition the electron must cross the forbidden region.

Semiclassical approximation

When the variation of the potential on the wavelenght is small, smaller then the energy itself, the situation is lose to the classical picture. In this case the quantum effects can be described by the semiclassical approximation, called also the WKB-approximation. In this approx the wave function is written as

$$\psi(x) = A(x) \exp\left(i \int_{x_0}^{x} p(x')\, dx' \right) \tag{9}$$

A central role plays the calssicalmomentum, which defines the main exponential factor, and also the pre-exponential coefficient

$$A(x) = \frac{C}{\sqrt{p(x)}} \tag{10}$$

(Though this coefficient would not be considered below in detail, beein approximated by a constant A(x)~A.)

Transition over the forbidden region

In order to estimate the descreas of the wave function when transition from the lower to the upper continuum takes place considered consider the semiclassical wave function, in which x_0 is a point in the lower continuuum (region one), and x is a point in the upper continuum (region two). It suffices to bring both these points to the boundaries of each region

$$\varepsilon + F x_0 = -m \quad (11)$$

$$\varepsilon + F x = m \quad (11)$$

Then with the exponential accuracy the amplitude of the transition is described by

$$\psi \simeq A \exp\left(i \int_{x_0}^{x} p(x')\, dx' \right) = A \exp\left(- \int_{x_0}^{x} \left| p(x') \right| dx' \right) \quad (12)$$

$$|p(x')| = (m^2 - (\varepsilon + F x')^2)^{1/2} = m\left(1 - \left(\frac{\varepsilon + F x'}{m}\right)^2\right)^{1/2} \quad (13)$$

Introduce y instead of x'

$$y = \frac{\varepsilon + F x'}{m},$$

$$dx' = \frac{m}{F} dy. \quad (14)$$

Then

$$\int_{x_0}^{x} |p(x')|\, dx' = m \int_{x_0}^{x} \left(1 - \left(\frac{\varepsilon + F x'}{m}\right)^2\right)^{1/2} dx' = \frac{m^2}{F} \int_{-1}^{1} (1 - y^2)^{1/2} dy$$

In[1]:=

$$\int_{-1}^{1} (1 - y^2)^{1/2} dy$$

Out[1]=

$$\frac{\pi}{2}$$

Thus

$$\int_{x_0}^{x} |p(x')|\, dx' = \frac{\pi}{2}\frac{m^2}{F}$$

Amplitude is

$$\psi \sim \exp\left(-\frac{\pi}{2}\frac{m^2}{F}\right) \tag{15}$$

Probability of the pair creation is

$$W = |\psi|^2 \sim \exp\left(-\pi\frac{m^2}{F}\right) = \exp\left(-\pi\frac{m^2}{eE}\right) = \exp\left(-\pi\frac{E_{QED}}{E}\right) \tag{16}$$

Here obviously $e, E > 0$, and

$$E_{QED} = \frac{m^2}{e} \equiv \frac{m^2 c^3}{\hbar e}$$

More accurately, the rate of the pair production per unit volume can be shown to be

$$\frac{dW}{dt\, d^3 r} = \frac{m^4}{4\pi^3}\left(\frac{E}{E_{QED}}\right)^2 \exp\left(-\pi\frac{E_{QED}}{E}\right) \equiv \frac{1}{4\pi^3}\left(\frac{mc^2}{\hbar}\right)\left(\frac{mc}{\hbar}\right)^3\left(\frac{E}{E_{QED}}\right)^2 \exp\left(-\pi\frac{E_{QED}}{E}\right) \tag{17}$$

It is valid provided

$$E \ll E_{QED} \tag{18}$$

Created by *Mathematica* (October 12, 2007)

Valid XHTML 1.1!