Casimir effect for metal plates

Casimir (1948) :

There is an attraction between two metal

plates which is due to the perturbation of

the photon vacuum produced by metal

plates.

The force per area is :

$$f = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}$$

Introduction

Assume there are two parallel metal plates (in the x-y plane) separated by a distance *a* in z-direction. In between the plates the photon momentum is quantized in the perpendicular direction. Energy and number of photon modes between the plates differ from their vacuum values

$$\mathrm{Sin}[k_z a] = 0;$$

$$k_z = \frac{\pi n}{a};$$

$$n = \pm 1, \pm 2 \ldots \qquad (1)$$

$$\omega = \sqrt{k^2 + \left(\frac{\pi n}{a}\right)^2}; \qquad (2)$$

$k = (k_x, k_y)$ - parallel momentum; (2)

Energy and force

$$\left(\frac{dE}{dV}\right)_{vac} = 2 \int \frac{d^2 k}{(2\pi)^2} \int \frac{dk_t}{2\pi} \frac{1}{2} \omega = \int \frac{d^2 k}{(2\pi)^2} \int \frac{dk_t}{2\pi} \omega; \qquad (3)$$

2 comes from two polirizations of photons. $\frac{1}{2}\omega$ is the contribution of each photon mode to the vacuum energy; k_t is the z – component of the momentum, $\mathbf{k} = (k_x, k_y)$, $\omega = \sqrt{k^2 + k_t^2}$.

Deriving $\left(\frac{dE}{dV}\right)_{vac}$ conventional running waves boundary conditions are assumed.

$$\left(\frac{dE}{dV}\right)_{met} = \frac{1}{2a} \sum_{n=-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2}\, \omega ; \qquad (4)$$

$\sum_{n=-\infty}^{\infty}$ is the summation over the discrete perpendicular momentum. For

running waves the summation reads $\frac{1}{a}\sum_{n=-\infty}^{\infty}$. However, for metals one needs to use zero boindary conditions, which means that standing waves are needed. Each standing wave corresponds to two running waves, thus giving a factor of

two. This fact needs to be compensated for by the explicit

factor 1 / 2 in front of the summation, resulting in $\frac{1}{2a}\sum_{n=-\infty}^{\infty}$.

Renormalization

$$\left(\frac{dE}{dV}\right)_{Ren} = \left(\frac{dE}{dV}\right)_{met} - \left(\frac{dE}{dV}\right)_{vac} = \int \frac{d^2 k}{(2\pi)^2} \left(\frac{1}{2a} \sum_{n=-\infty}^{\infty} \sqrt{k^2 + \left(\frac{\pi n}{a}\right)^2} - \int_{-\infty}^{\infty} \frac{dk_t}{2\pi} \sqrt{k^2 + k_t^2}\right); \quad (5)$$

Scale k_t, $k_t \to \nu$, $k_t = \frac{\nu\pi}{a}$ (6)

$$\left(\frac{dE}{dV}\right)_{Ren} = \frac{1}{2a} \int \frac{d^2 k}{(2\pi)^2} \left(\sum_{n=-\infty}^{\infty} \sqrt{k^2 + \left(\frac{\pi n}{a}\right)^2} - \int_{-\infty}^{\infty} d\nu \sqrt{k^2 + \left(\frac{\pi \nu}{a}\right)^2}\right) \quad (6)$$

Simplify notation, identifying in the integral $\nu = n$ (7)

$$\left(\frac{dE}{dV}\right)_{Ren} = \frac{1}{2a} \int \frac{d^2 k}{(2\pi)^2} \left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right) \sqrt{k^2 + \left(\frac{\pi n}{a}\right)^2} = \frac{1}{2a} \int \frac{d^2 k}{(2\pi)^2} \left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right) \omega; \quad (7)$$

$$\omega = \sqrt{k^2 + \left(\frac{\pi n}{a}\right)^2} \quad (8)$$

Energy per area A is (9)

$$\mathcal{E} \equiv \frac{dE}{dA} = a\,\frac{dE}{dV} = \frac{1}{2}\int \frac{d^2 k}{(2\pi)^2}\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)\omega; \quad (9)$$

Force per are (10)

$$f = -\frac{\partial \mathcal{E}}{\partial a} = \frac{1}{2}\int \frac{d^2 k}{(2\pi)^2}\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)\frac{(\pi n)^2}{a^3}\,\frac{1}{\omega} \quad (10)$$

$$= \frac{1}{8a^3}\int d^2 k\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)\frac{n^2}{\omega}; \quad (10)$$

Scaling f =-Const/a^4

$$d^2 k = 2\pi k\, dk = \pi\, d(k^2);$$ (11)

$$f = \frac{\pi}{8a^3}\int_0^\infty d(k^2)\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)\frac{n^2}{\omega} = \frac{\pi}{4a^3}\int_0^\infty d(k^2)\left(\sum_{n=1}^{\infty} - \int_{0}^{\infty} dn\right)\frac{n^2}{\omega};$$ (12)

$$k^2 = \left(\frac{\pi}{a}\right)^2 \xi; \quad \omega = \sqrt{k^2 + \left(\frac{\pi n}{a}\right)^2} = \frac{\pi}{a}\sqrt{\xi + n^2};$$ (13)

$$f = \frac{\pi}{4a^3}\left(\frac{\pi}{a}\right)^{2-1}\int_0^\infty d\xi\left(\sum_{n=1}^{\infty} - \int_{0}^{\infty} dn\right)\frac{n^2}{\sqrt{\xi + n^2}} = \frac{\pi^2}{4a^4}\int_0^\infty d\xi\left(\sum_{n=1}^{\infty} - \int_{0}^{\infty} dn\right)\frac{n^2}{\sqrt{\xi + n^2}};$$ (14)

$$f = -\frac{\text{Const}}{a^4};$$
$$\text{Const} = \frac{\pi^2}{4}\int_0^\infty d\xi \left(\int_0^\infty dn - \sum_{n=1}^\infty\right) \frac{n^2}{\sqrt{\xi + n^2}};$$ **(15)**

Calculation of Const

Cutoff with the help of Gamma function

$$\Gamma(n) = \int_0^\infty \tau^{n-1}\,\text{Exp}[-\tau]\,d\tau;$$ **(16)**

$$\tau = z\,t;$$ **(16)**

$$\Gamma(n) = z^n \int_0^\infty t^{n-1}\, \mathrm{Exp}[-z\,t]\, dt; \qquad (16)$$

$$\frac{1}{z^n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1}\, \mathrm{Exp}[-z\,t]\, dt; \qquad (16)$$

$$\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Exp}[-z\,t]\, \frac{dt}{\sqrt{t}}; \qquad \left(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right); \qquad (16)$$

$$\frac{1}{\sqrt{\xi + n^2}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Exp}[-(\xi + n^2)\,t]\, \frac{dt}{\sqrt{t}}; \qquad (17)$$

$$\mathrm{Const} = \frac{\pi^2}{4} \int_0^\infty d\xi \left(\int_0^\infty dn - \sum_{n=1}^{\infty} \right) \frac{n^2}{\sqrt{\xi + n^2}}; \qquad (18)$$

$$\int_0^\infty d\xi\, \mathrm{Exp}[-\xi\, t] = \frac{1}{t}; \qquad (19)$$

$$\text{Const} = \frac{\pi^2}{4}\,\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{dt}{t^{3/2}}\left(\int_0^\infty dn - \sum_{n=1}^{\infty}\right) n^2\,\text{Exp}[-n^2 t];$$ **(20)**

$$\text{Const} = \frac{\pi^{3/2}}{4}\int_0^\infty \frac{dt}{t^{3/2}}\,f[t];$$ **(21)**

$$f[t] = \left(\int_0^\infty dn - \sum_{n=1}^{\infty}\right) n^2\,\text{Exp}[-n^2 t];$$ **(21)**

Numerical calculations

$$\text{Const} = \frac{\pi^{3/2}}{4}\int_0^\infty \frac{f[t]}{t^{3/2}}\,dt;$$ **(22)**

$$f[t] = \left(\int_0^\infty dn - \sum_{n=1}^\infty \right) n^2 \, \text{Exp}[-n^2 t]; \qquad \textbf{(22)}$$

$$\int_0^\infty dn \, n^2 \, \text{Exp}[-n^2 t] = -\frac{\partial}{\partial t} \int_0^\infty dn \, \text{Exp}[-n^2 t] = -\frac{\partial}{\partial t} \frac{1}{2} \sqrt{\frac{\pi}{t}} = \frac{\sqrt{\pi}}{4 t^{3/2}}; \qquad \textbf{(22)}$$

$$f[t\_] := \frac{\sqrt{\pi}}{4 t^{3/2}} - \sum_{n=1}^\infty n^2 \, \text{Exp}[-n^2 t]; \qquad (23)$$

$$\text{Plot}\left[\frac{f[t]}{t^{3/2}}, \{t, .3, 10\}, \text{Frame} \to \text{True}, \text{FrameLabel} \to \{t, f[t]/t^{3/2}\}, \text{DefaultFont} \to \text{Times}[20] \right] \qquad (23)$$

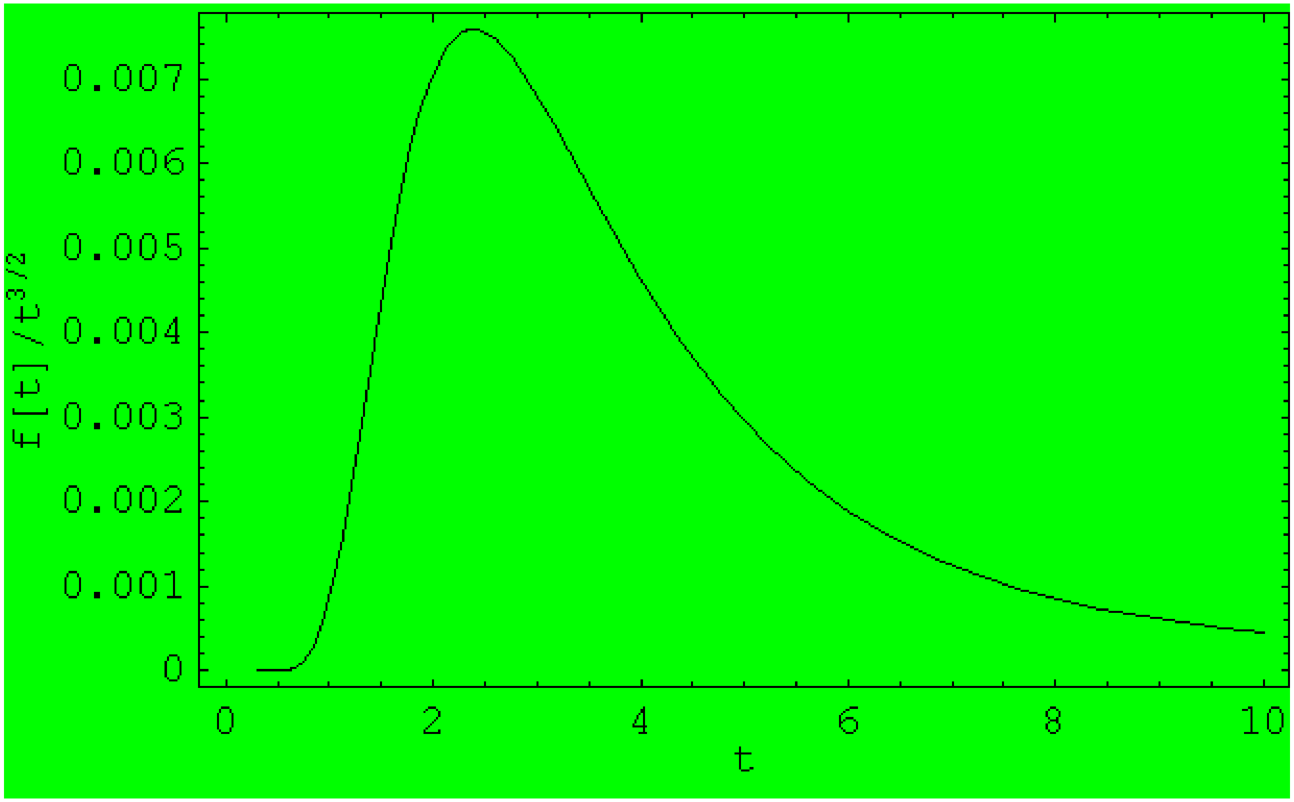

Figure 1

- Graphics -

$$\text{Const} = \frac{\pi^{3/2}}{4} \int_{0.05}^{\infty} \frac{f[t]}{t^{3/2}}\, dt \;//\; N = 0.04112334; \tag{24}$$

Analytical calculation (presented below) gives **(25)**

$$\text{Const} = \frac{\pi^2}{240} \tag{25}$$

$(240/\pi^2)$ Const

1.

Analytical calculation

$$\text{Const} = \frac{\pi^{3/2}}{4} \int_0^\infty \frac{dt}{t^{3/2}} \left(\int_0^\infty dn - \sum_{n=1}^{\infty}\right) n^2\, \text{Exp}[-n^2 t]; \tag{26}$$

$$\text{Const} = \frac{\pi^{3/2}}{4}\int_0^\infty \frac{dt}{t^{3/2}}\left(-\frac{d}{dt}\right)\left(\int_0^\infty dn - \sum_{n=1}^{\infty}\right)\text{Exp}[-n^2\,t] =$$
$$\frac{\pi^{3/2}}{8}\int_0^\infty \frac{dt}{t^{3/2}}\left(-\frac{d}{dt}\right)\left(\int_{-\infty}^\infty dn - \sum_{n=-\infty}^{\infty}\right)\text{Exp}[-n^2\,t] =$$
$$\frac{\pi^{3/2}}{8}\int_0^\infty \frac{dt}{t^{3/2}}\left(-\frac{d}{dt}\right)\left(\sqrt{\frac{\pi}{t}} - \sum_{n=-\infty}^{\infty}\text{Exp}[-n^2\,t]\right) \quad \textbf{(26)}$$

Poisson summation formula

Let **(27)**

$$\tilde{f}[k] = \int_{-\infty}^{\infty} dx\, f[x]\,\text{Exp}[i\,k\,x] \quad \textbf{(27)}$$

Then the Poisson summation formula is valid **(28)**

$$\sum_{n=-\infty}^{\infty} f[n] = \sum_{m=-\infty}^{\infty}\tilde{f}[2\,\pi\,m] \quad \textbf{(28)}$$

Example (29)

$$f[x] = \text{Exp}[-t\,x^2] \quad (29)$$

$$\tilde{f}[k] = \int_{-\infty}^{\infty} dx\, f[x]\, \text{Exp}[-t\,x^2 + i\,k\,x] = \sqrt{\frac{\pi}{t}}\, \text{Exp}\left[-\frac{k^2}{4\,t}\right] \quad (29)$$

$$\sum_{n=-\infty}^{\infty} \text{Exp}[-t\,n^2] = \sqrt{\frac{\pi}{t}} \sum_{m=-\infty}^{\infty} \text{Exp}\left[-\frac{\pi^2\,m^2}{t}\right] \quad (29)$$

Using the Poisson summation one gets (30)

$$\text{Const} = \frac{\pi^{3/2}}{8} \int_0^{\infty} \frac{dt}{t^{3/2}} \left(-\frac{d}{dt}\right) \left(\sqrt{\frac{\pi}{t}} - \sum_{n=-\infty}^{\infty} \text{Exp}[-n^2\,t]\right) =$$

$$\text{Const} =$$

$$\frac{\pi^{3/2}}{8} \int_0^{\infty} \frac{dt}{t^{3/2}} \left(-\frac{d}{dt}\right) \left(\sqrt{\frac{\pi}{t}} - \sqrt{\frac{\pi}{t}} \sum_{m=-\infty}^{\infty} \text{Exp}\left[-\frac{\pi^2\,m^2}{t}\right]\right) =$$

∞

$$\frac{\pi^{3/2}}{8}\int_0 \frac{dt}{t^{3/2}}\left(-\frac{d}{dt}\right)\left(-\sqrt{\frac{\pi}{t}}\sum_{m\neq 0}\mathrm{Exp}\left[-\frac{\pi^2 m^2}{t}\right]\right) =$$

$$\frac{\pi^2}{8}\,2\int_0^\infty \frac{dt}{t^{3/2}}\frac{d}{dt}\left(\sqrt{\frac{1}{t}}\sum_{m=1}^\infty\mathrm{Exp}\left[-\frac{\pi^2 m^2}{t}\right]\right) =$$

$$\frac{\pi^2}{8}\,2\int_0^\infty dt\,\sqrt{\frac{1}{t}}\sum_{m=1}^\infty\mathrm{Exp}\left[-\frac{\pi^2 m^2}{t}\right]\left(-\frac{d}{dt}\right)\frac{1}{t^{3/2}} = \tag{30}$$

$$\frac{\pi^2}{8}\,2\,\frac{3}{2}\int_0^\infty dt\,\sqrt{\frac{1}{t}}\sum_{m=1}^\infty\mathrm{Exp}\left[-\frac{\pi^2 m^2}{t}\right]\frac{1}{t^{5/2}} =$$

$$\frac{3\pi^2}{8}\int_0^\infty dt\,\frac{1}{t^3}\sum_{m=1}^\infty\mathrm{Exp}\left[-\frac{\pi^2 m^2}{t}\right] \underset{t=1/x}{=}$$

$$\frac{3\,\pi^2}{8}\sum_{m=1}^{\infty}\int_0^{\infty} dx\; x\,\mathrm{Exp}[-\pi^2\,m^2\,x] =$$

$$\frac{3\,\pi^2}{8}\sum_{m=1}^{\infty}\frac{1}{(\pi^2\,m^2)^2} = \frac{3}{8\,\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^4}$$

$$\mathrm{Const} = \frac{3}{8\,\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^4}\,; \qquad \textbf{(31)}$$

$$\sum_{m=1}^{\infty}\frac{1}{m^4} = \zeta[4] = \frac{\pi^4}{90}\,; \qquad \textbf{(32)}$$

Zeta[4]

$$\frac{\pi^4}{90}$$

$$\mathrm{Const} = \frac{3}{8\,\pi^2}\,\frac{\pi^4}{90} = \frac{\pi^2}{240}\,; \qquad \textbf{(33)}$$

$$\text{Const} = \frac{\pi^2}{240} \tag{34}$$

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