VII. Potentials gauge invariance, current conservation law,

The second pair of Maxwell's equations, which do not depend on charges and currents,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \textbf{(7.1)}$$

can be satisfied if

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad \textbf{(7.2)}$$

A transformation of the potentials,

$$\begin{aligned} V &\to V' = V - \dot{\alpha} \\ \mathbf{A} &\to \mathbf{A}' = \mathbf{A} + \nabla \alpha \end{aligned} \qquad \textbf{(7.3)}$$

called the gauge transformation, leaves fields same, i. e. $\mathbf{E}' = \mathbf{E}$, $\mathbf{B}' = \mathbf{B}$.

Exercise: Verify that the fields are gauge invariant.
Solution:

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla \alpha) = \nabla \times \mathbf{A} = \mathbf{B}$$

$$\mathbf{E}' = -\nabla V' - \dot{\mathbf{A}}' = -\nabla (V - \dot{\alpha}) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla \alpha) = -\nabla V - \frac{\partial}{\partial t}\mathbf{A} = \mathbf{E}$$

Among the four functions $V(\mathbf{r},t)$, $\mathbf{A}(\mathbf{r},t)$ only three are independent. Therefore one can choose one arbitrary condition on them, which is called the gauge condition.

Examples of gauge conditions:

- The Lorentz gauge

$$\frac{1}{c^2}\dot{V} + \nabla \cdot \mathbf{A} = 0 \qquad \textbf{(7.4)}$$

Dimensions: $[energy] = [e][V]$, $[momentum] = [e][A]$, which means

$$[V] = [velocity][A] \qquad \textbf{(7.5)}$$

From Eq.(7.5) it follows that dimensions in Eq. (7.4) are correct.

Comment. In 4D notation the Lorents gauge Eq.(7.4) reads

$$\partial_\mu A^\mu = 0 \qquad \textbf{(7.6)}$$

Comapare the current conservation law Eq.(6.6)

- The Coulomb gauge

$$\nabla \cdot \mathbf{A} \qquad \textbf{(7.7)}$$

- The axial gauge

$$\mathbf{a} \cdot \mathbf{A} = 0 \qquad \textbf{(7.8)}$$

where **a** is an arbitrary constant vector (for example $\mathbf{a} = (0,0,1)$ **).**

Substituting Eqs. (7.2) into the Maxwell's equations one finds the equations on the potentials. One of them follows from the Gauss law

$$\nabla \cdot (-\nabla V - \dot{\mathbf{A}}) = \frac{1}{\varepsilon_0}\rho \qquad \textbf{(7.9)}$$

This gives

$$-\Delta V - \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{\varepsilon_0}\rho \qquad \textbf{(7.10)}$$

In the Coulomb gauge Eq.(7.4) this leads to

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)V = \frac{1}{\varepsilon_0}\rho \qquad \textbf{(7.11)}$$

From the Ampere (+Maxwell's term) law one derives

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) + \mu_0 \mathbf{j} \qquad \textbf{(7.12)}$$

Since $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$, Eq. (7.12) ca be rewrittern as

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A} + \frac{1}{c^2}\nabla\dot{V} + \nabla\left(\nabla \cdot \mathbf{A}\right) = \mu_0 \mathbf{j} \qquad \textbf{(7.13)}$$

In the Lorentz gauge this simplifies to

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A} = \mu_0 \mathbf{j} \qquad \textbf{(7.14)}$$

Thus, Eqs.(7.11),(7.14) for the potentials read

$$\begin{aligned} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)V(\mathbf{r},t) &= \frac{1}{\varepsilon_0}\rho(\mathbf{r},t) \\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A}(\mathbf{r},t) &= \mu_0 \ \mathbf{j}(\mathbf{r},t) \end{aligned} \qquad \textbf{(7.15)}$$

Compare Eqs.(4.3) for the static case.

Differentiating Eqs.(7.15) and using the current conservation law Eq.(6.3) one derives

$$\begin{aligned} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\left(\frac{1}{c^2}\frac{\partial}{\partial t}V + \nabla \cdot \mathbf{A}\right) &= \frac{1}{c^2\,\varepsilon_0}\frac{\partial}{\partial t}\rho + \mu_0 \nabla \cdot \mathbf{j} \\ &= \mu_0\left(\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j}\right) = 0 \end{aligned} \qquad \textbf{(7.16)}$$

In other words,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\left(\frac{1}{c^2}\frac{\partial}{\partial t}V+\nabla\cdot\mathbf{A}\right)=0 \qquad \textbf{(7.17)}$$

which complies with the Lorentz gauge Eq.(7.4).

The Lorentz gauge was chosen using the gauge invariance. The argument can be reversed. The current conservation law can be considered as a foundation for the gauge invariance.

Comment.

The operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta$ is often called as ∂^2. There are several way to present it

$$\partial^2=\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}=\partial_\mu\partial^\mu=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta=-\Box \qquad (7.18)$$

People also use notation

$$\Box=\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad (7.19)$$

The operator $\Box$ is called the D'Alamberian.

Notation becomes more palatable if one presumes that along with the *contravariant* four vector C^μ,

$$C^\mu=\left(C^0,\mathbf{C}\right) \qquad (7.20)$$

there exists also the *covariant* vector C_μ defined as

$$C_\mu=\left(C^0,-\mathbf{C}\right) \qquad (7.21)$$

The scalar product of the two four-vectors C^μ, D^μ is written as

$$\begin{aligned}C^\mu D_\mu&=\sum_{\mu=0}^{3}C^\mu D_\mu=C^0D_0+C^1D_1+C^2D_2+C^3D_3=\\&=C^0D^0-C^1D^1-C^2D^2-C^3D^3=C^0D^0-\mathbf{C}\cdot\mathbf{D}\end{aligned} \qquad (7.22)$$

In particular

$$\begin{aligned}x^\mu&=(ct,\ \mathbf{r})\\x_\mu&=(ct,-\mathbf{r})\end{aligned} \qquad (7.23)$$

Eqs.(7.23) implies

$$\begin{aligned}\partial^\mu&=\frac{\partial}{\partial x_\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right)\\\partial_\mu&=\frac{\partial}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\ \ \nabla\right)\end{aligned} \qquad (7.24)$$

Notice the "opposite" signs in Eqs. (7.23),(7.24).

Exercise:

- verify that (7.24) complies with (7.23).
- verify that (6.6) complies with (7.23).

Eq.(4.7) shows that Eqs.(7.15) can be written as one equation

$$\partial^2 A^\mu = \mu_0 \, j^\mu \tag{7.25}$$

For the static case, when $A^\mu = A^\mu(\mathbf{r})$, Eq.(7.25) reduces to (4.8). Clearly, the Lorentz gauge Eq.(7.6) and the equation of moton (7.25) comply with the current conservation law Eq.(6.6).

Summary:

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A}} \tag{7.26}$$

$$\boxed{\begin{aligned} &V \to V' = V - \dot{\alpha} \\ &\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \alpha \\ &\mathbf{E}' = \mathbf{E} \\ &\mathbf{B}' = \mathbf{B} \end{aligned}} \tag{7.27}$$

In the Lorentz gauge the potentials satisfy

$$\boxed{\begin{aligned} &\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) V(\mathbf{r},t) = \frac{1}{\varepsilon_0}\rho(\mathbf{r},t) \\ &\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) \mathbf{A}(\mathbf{r},t) = \mu_0 \, \mathbf{j}(\mathbf{r},t) \end{aligned}} \tag{7.28}$$

or

$$\boxed{\partial^2 A^\mu = \mu_0 \, j^\mu} \tag{7.29}$$