VI. Maxwell's equations

The Ampere's law written in the form

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{j} \qquad \textbf{(6.1)}$$

implies that the current conservation law reads

$$\nabla \cdot \mathbf{j} = 0 \qquad \textbf{(6.2)}$$

Eq. (6.2) is true for steady currents, but is wrong in general time-dependent case.

The correct expression for the current conservation in general case reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad \textbf{(6.3)}$$

In order to fix the problem with the current conservation Maxwell suggested that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0}\rho & \qquad \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \, \mathbf{j} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \qquad \nabla \cdot \mathbf{B} &= 0 \end{aligned} \qquad \textbf{(6.4)}$$

From

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \, \mathbf{j} \qquad \textbf{(6.5)}$$

one finds

$$0 = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla \cdot \mathbf{E} + \mu_0 \nabla \cdot \mathbf{j} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right) + \mu_0 \nabla \cdot \mathbf{j} = \mu_0 (\dot{\rho} + \nabla \cdot \mathbf{j})$$

which reproduces the correct current conservation law Eq.(6.3). Here the Gauss law $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$ was used.

The electric field in general case depends on the variation of the vector potential in time.

Comment:
The current conservation law can be conveniently written as

$$\frac{\partial j^\mu}{\partial x^\mu} = 0 \qquad \textbf{(6.6)}$$

if one uses the four-vectors for the current j^μ defined in Eq.(4.7) and introduces the four vector of coordinates x^μ

$$x^\mu = (ct, \mathbf{r}) \qquad \textbf{(6.7)}$$

It is presumed that there is the summation over the repeated Greek index, i. e.

$$\frac{\partial j^{\mu}}{\partial x^{\mu}} = \sum_{\mu=0}^{4} \frac{\partial j^{\mu}}{\partial x^{\mu}} = \frac{\partial j^{0}}{\partial x^{0}} + \frac{\partial j^{1}}{\partial x^{1}} + \frac{\partial j^{2}}{\partial x^{2}} + \frac{\partial j^{3}}{\partial x^{3}} = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad \textbf{(6.8)}$$

Summary:

$$\begin{array}{ll} \nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}\rho & \nabla \times \mathbf{B} = \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} + \mu_0\, \mathbf{j} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \nabla \cdot \mathbf{B} = 0 \end{array} \qquad \textbf{(6.9)}$$

$$\begin{array}{l} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \\ \displaystyle\oint_S \mathbf{j} \cdot \mathbf{dS} = -\frac{\partial Q}{\partial t}, \quad Q = \int_V \rho\, d^3 r \end{array} \qquad \textbf{(6.10)}$$

Maxwell, James Clerk (1831-1879), Scottish mathematical physicist. The greatest work of Maxwell's life was devoted to electricity. Maxwell's most important contribution was the extension and mathematical formulation of earlier work of Ampère, Faraday, and others into a linked set of equations (originally, 20 equations in 20 variables, later re-expressed in quaternion and vector-based notations). These equations, were first presented to the Royal Society in 1864. Maxwell showed that the equations predict waves of oscillating electric and magnetic fields that travel through empty space at a speed that could be predicted from simple electrical experiments—using the data available at the time, Maxwell obtained a velocity of 310,740,000 m/s. Maxwell (1865) wrote:

This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.

VII. Potentials gauge invariance, current conservation law,

The second pair of Maxwell's equations, which do not depend on charges and currents,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0 \tag{7.1}$$

can be satisfied if

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{7.2}$$

A transformation of the potentials,

$$\begin{aligned} V \ &\rightarrow V' \ = V - \dot{\alpha} \\ \mathbf{A} &\rightarrow \mathbf{A}' = \mathbf{A} + \nabla \alpha \end{aligned} \tag{7.3}$$

called the gauge transformation, leaves fields same, i. e. $\mathbf{E}' = \mathbf{E}$, $\mathbf{B}' = \mathbf{B}$.

Exercise: Verify that the fields are gauge invariant.
Solution:

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla \alpha) = \nabla \times \mathbf{A} = \mathbf{B}$$

$$\mathbf{E}' = -\nabla V' - \dot{\mathbf{A}}' = -\nabla (V - \dot{\alpha}) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla \alpha) = -\nabla V - \frac{\partial}{\partial t}\mathbf{A} = \mathbf{E}$$

Among the four functions $V(\mathbf{r}, t)$, $\mathbf{A}(\mathbf{r}, t)$ only three are independent. Therefore one can choose one arbitrary condition on them, which is called the gauge condition.

Examples of gauge conditions:

- The Lorentz gauge

$$\frac{1}{c^2}\dot{V} + \nabla \cdot \mathbf{A} = 0 \tag{7.4}$$

Dimensions: $[energy] = [e][V]$, $[momentum] = [e][A]$, which means

$$[V] = [velocity][A] \tag{7.5}$$

From Eq.(7.5) it follows that dimensions in Eq. (7.4) are correct.

Comment. In 4D notation the Lorents gauge Eq.(7.4) reads

$$\partial_{\mu} A^{\mu} = 0 \tag{7.6}$$

Comapare the current conservation law Eq.(6.6)

- The Coulomb gauge

$$\nabla \cdot \mathbf{A} \tag{7.7}$$

- The axial gauge

$$\mathbf{a} \cdot \mathbf{A} = 0 \tag{7.8}$$