III. Magnetostatic

Maxwell's equations

$$\nabla\times\mathbf{B}=\mu_0\,\mathbf{j} \qquad \textbf{(3.1)}$$

$$\nabla\cdot\mathbf{B}=0 \qquad \textbf{(3.2)}$$

$$\mu_0=4\pi 10^{-7} \qquad \textbf{(3.3)}$$

Vector potential **A**

$$\mathbf{B}=\nabla\times\mathbf{A} \qquad \textbf{(3.4)}$$

Useful identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad \textbf{(3.5)}$$

Eq.(3.5) implies

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\Delta\mathbf{A} \qquad \textbf{(3.6)}$$

We will see below that

$$\nabla\cdot\mathbf{A}=0 \qquad \textbf{(3.7)}$$

Eqs.(3.6),(3.7) give

$$\nabla\times(\nabla\times\mathbf{A})=-\Delta\mathbf{A} \qquad \textbf{(3.8)}$$

Eqs.(3.1),(3.4),(3.8) result in

$$\Delta\mathbf{A}=-\mu_0\mathbf{j} \qquad \textbf{(3.9)}$$

Taking divergence in Eq.(3.9) one finds

$$\Delta(\nabla\cdot\mathbf{A})=-\mu_0(\nabla\cdot\mathbf{j}) \qquad \textbf{(3.10)}$$

The current conservation law

$$\boxed{\nabla\cdot\mathbf{j}=0} \qquad \textbf{(3.11)}$$

Eqs.(2.15),(2.16) show that Eq.(3.9) can be rewritten as

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \qquad \textbf{(3.12)}$$

The important theorem:

$$\int_S\nabla\times\mathbf{W}\,dS=\int_L\mathbf{W}\cdot\mathbf{dl} \qquad \textbf{(3.13)}$$

Here *S* and *L* are the surface (not closed) and the contour, which restricts this surface.
Eqs.(3.1),(3.13) give

$$\boxed{\int_L\mathbf{B}\cdot\mathbf{dl}=\mu_0\int_S\mathbf{j}\cdot\mathbf{dS}=\mu_0 J} \qquad \textbf{(3.14)}$$

Here *J* is the total current that crosses the surface *S*.

Summary:

The Maxwell's equations

$$\boxed{\begin{aligned}\nabla\cdot\mathbf{B}&=0\\ \nabla\times\mathbf{B}&=\mu_0\,\mathbf{j}\end{aligned}} \qquad \textbf{(3.15)}$$

give **one** equation for the vector potential **A**

$$\boxed{\Delta\mathbf{A}(\mathbf{r})=-\mu_0\mathbf{j}(\mathbf{r})} \qquad \textbf{(3.16)}$$

which can be presented also as

$$\boxed{\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'} \qquad \textbf{(3.17)}$$

IV. Review of electrostatic + magnetostatic

Maxwell's equations

$$\nabla\times\mathbf{E}=0 \qquad \nabla\cdot\mathbf{B}=0$$
$$\nabla\cdot\mathbf{E}=\frac{1}{\varepsilon_0}\rho \qquad \nabla\times\mathbf{B}=\mu_0\mathbf{j} \tag{4.1}$$

Potentials $V,\mathbf{A}$

$$\mathbf{E}=-\nabla V \qquad \mathbf{B}=\nabla\times\mathbf{A} \tag{4.2}$$

satisfy

$$\Delta V(\mathbf{r})=-\frac{1}{\varepsilon_0}\rho(\mathbf{r})$$
$$\Delta\mathbf{A}(\mathbf{r})=-\mu_0\mathbf{j}(\mathbf{r}) \tag{4.3}$$

Introduce the four-vector potential A^μ, and the four-vector current j^μ (density of current, actually, but usually it is called simply the current)

$$A^\mu=\left(\frac{1}{c}V,\mathbf{A}\right), \qquad \mu=0,1,2,3$$
$$j^\mu=\left(c\rho,\ \mathbf{j}\ \right), \tag{4.4}$$

Verify dimensions in Eqs.(4.4).

$$\begin{cases}[\mathrm{E}]=\dfrac{[\mathrm{V}]}{[\mathrm{meter}]}, \quad [\mathrm{B}]=\dfrac{[\mathrm{A}]}{[\mathrm{meter}]}\\ [\mathrm{E}]=[\mathrm{velocity}][\mathrm{B}]\\ \dfrac{1}{[\mathrm{velocity}]}[\mathrm{V}]=[\mathrm{A}]\end{cases} \tag{4.5}$$

The last line in Eqs.(4.5) shows that all four components in A^μ have same dimension, $[A^\mu]=[\mathbf{A}]$. From

$$[j]=[velocity][\rho] \tag{4.6}$$

one sees that all four components of the current also have same dimension, $[j^\mu]=[\mathbf{j}]$.

Conclusion: dimensions in Eqs.(4.4) are OK.

Notation introduced in Eqs. (4.4) simply means that

$$A^0=\frac{1}{c}V,\ A^1=\mathbf{A}_x,\ \ A^2=\mathbf{A}_y,\ \ A^3=\mathbf{A}_z$$
$$j^0=c\rho,\ \ j^1=\mathbf{j}_x,\ \ j^2=\mathbf{j}_y,\ \ j^3=\mathbf{j}_z \tag{4.7}$$

This notation allows to write Eqs.(4.3) as one equation for the four-vectors

$$\boxed{\Delta A^{\mu}(\mathbf{r}) = -\mu_0\, j^{\mu}(\mathbf{r})} \qquad \textbf{(4.8)}$$

Verifying the case $\mu = 0$ remember that

$$c = \frac{1}{\sqrt{\varepsilon_0\,\mu_0}} = \sqrt{9\cdot 10^9 \cdot 10^7} = 3\cdot 10^8\, m/s \qquad \textbf{(4.9)}$$

is the velocity of light.

Remember Eqs.(4.3)

$$V(\mathbf{r}) = \frac{1}{4\pi\,\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3 r'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3 r'$$

Eqs. (4.4) allow to present these equations as one equation for four-vectors

$$\boxed{A^{\mu}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{j^{\mu}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3 r'} \qquad \textbf{(4.10)}$$

Comment.

Eq. (4.10) is written for the static case, but it provides a *very* convenient opportunity to describe the general case. When the current and potential are time-dependent, i.e. $A^{\mu}(\mathbf{r},t),\ j^{\mu}(\mathbf{r},t)$, then it turns out (see below) that Eq. (4.10) should be modified as follows

$$A^{\mu}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{j^{\mu}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\, d^3 r' \qquad \textbf{(4.11)}$$

where t' is the time in the “past”, which satisfies the “propagation condition”

$$t - t' = \frac{|\mathbf{r}-\mathbf{r}'|}{c} \qquad \textbf{(4.12)}$$

being thus a function of $\mathbf{r}', \mathbf{r},\ t$. In the conventional 3D notation Eq.(4.11) reads

$$V(\mathbf{r},t) = \frac{1}{4\pi\,\varepsilon_0}\int \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\, d^3 r'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\, d^3 r' \qquad \textbf{(4.13)}$$