Solution of the equation $\partial^2 f = Q$.

Consider the differential equation for the function $f(\mathbf{r},t)$

$$\left(\frac{1}{c^2}\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \Delta\, f(\mathbf{r},t)\right) = Q(\mathbf{r},t) \tag{1.1}$$

where $Q(\mathbf{r},t)$ is a known function. Let us verify that a solution of this equation can be written in the form

$$f(\mathbf{r},t) = \frac{1}{4\pi}\int \frac{Q(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \tag{1.2}$$

where

$$t' = t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'| \tag{1.3}$$

Remember firstly that if $g(\mathbf{r})$ satisfies the equation

$$-\Delta V(\mathbf{r}) = \rho(\mathbf{r}) \tag{1.4}$$

then

$$V(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' \tag{1.5}$$

Eq.(1.5) can be thought of as the Coulomb law for the potential $V(\mathbf{r})$ produced by the charge distribution $\rho(\mathbf{r})$. For this interpretation Eq.(1.4) represents the Gauss's law $\nabla\cdot\mathbf{E} = \rho$, in which $\mathbf{E} = -\nabla V$.

Similarly it is convenient to think of the function $Q(\mathbf{r},t)$ in Eq.(1.1) as a charge density. Let us prove validity of Eq.(1.2) for the special case, when $Q(\mathbf{r},t)$ in Eq.(1.1) represents a point-like charge, which is located at $\mathbf{r}_0$ at the moment of time *t*.
Then Eq.(1.1) can be written in a form

$$\frac{1}{c^2}\frac{\partial^2 f(\mathbf{r},t)}{\partial t^2} - \Delta\, f(\mathbf{r},t) = 0 \tag{1.6}$$

everywhere, except for a small region where the charge Q is actually located at the moment *t*, i.e. in all the space except for $|\mathbf{r}-\mathbf{r}_0| = 0$. It is convenient to change variables. Instead of **r** introduce **R,**

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0 \tag{1.7}$$

Since the function Q represents a point-like charge we can presume that $\delta f(\mathbf{r},t) \equiv \delta f(R,t)$. As a result the Laplacian simplifies to be

$$\Delta_{\mathbf{r}}\, f(R,t) = \Delta_{\mathbf{R}}\, f(R,t) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\, f(R,t)\right) \tag{1.8}$$

It can be further simplified if we introduce a new function $\delta F(R,t)$,

$$f(R,t) = \frac{F(R,t)}{R} \tag{1.9}$$

because it is easy to verify that

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}f(R,t)\right)=\frac{1}{R}\frac{\partial^2}{\partial R^2}F(R,t) \qquad (1.10)$$

In these variables Eq.(1.6) reads

$$\frac{1}{c^2}\frac{\partial^2 F(R,t)}{\partial t^2}-\frac{\partial^2 F(R,t)}{\partial R^2}=0 \qquad (1.11)$$

A solution of this equation can be written in the form

$$F(R,t)=G\left(t-\frac{R}{c}\right) \qquad (1.12)$$

which implies that

$$f(R,t)=\frac{1}{R}G\left(t-\frac{R}{c}\right) \qquad (1.13)$$

where $G(x)$ is an arbitrary function. This solution is applicable everywhere, except for a close vicinity of the origin $R=0$. Consider now what happens in this particular region, i.e. take $R\to 0$. Here the charge density in δQ and the Laplacian become very large (see 1/R factor in Eq.(1.13)). As a result, the term with the time derivatives becomes irrelevant in this region, and Eq.(1.1) reads

$$-\Delta f = Q, \quad \text{when } R \text{ is small} \qquad (1.14)$$

The solution of the latter equation reads (compare Eq.(1.5))

$$f(R,t)=\frac{1}{4\pi}\int\frac{Q(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r' \qquad (1.15)$$

where the integration is located in very close vicinity of the $\mathbf{r}_0$, where the charge is located. The latter fact allows oneto rewrite Eq.(1.15)

$$f(R,t)=\frac{1}{4\pi}\frac{1}{R}\int Q(\mathbf{r}',t)\,d^3r' \qquad (1.16)$$

Comparing this equation with Eq.(1.13) one finds that the latter equation can be extended into the region of small distances $R\to 0$, provided the function *G* satisfies

$$G(t)=\frac{1}{4\pi}\int Q(\mathbf{r}',t)\,d^3r' \qquad (1.17)$$

From Eqs.(1.13), (1.17) we deduce that a solution of Eq.(1.6), which is valid everywhere, reads

$$f(\mathbf{r},t)=\frac{1}{4\pi}\frac{1}{R}\int Q\left(\mathbf{r}',t-\frac{R}{c}\right)d^3r'=\frac{1}{4\pi}\int\frac{1}{R}Q\left(\mathbf{r}',t-\frac{R}{c}\right)d^3r' \qquad (1.18)$$

The latter identity holds because we can use again the fact that the integration is saturated in close vicinity of the $\mathbf{r}_0$, where the charge is located.

Eq.(1.18) justifies (1.2) for the case of one charge. Similar result holds for an arbitrary system of charges, and therefore for an arbitrary charge distribution.