EM waves

Vacuum

Wave equation
1D
Monochromatic wave
Polarization
Energy, momentum

Matter (insulators)

Wave equation
Dispersion

Reflection

Polarization
Brewster's angle

EM Waves in Vacuum

Wave eq for $\vec{E}, \vec{B}$

$$\rho = \vec{J} = 0 \qquad (1)$$

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 0 \\ \vec{\nabla} \times \vec{E} = - \dot{\vec{B}} \end{cases} \quad \begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \dot{\vec{E}} \end{cases} \qquad (2)$$

$$\underbrace{\nabla \times \left(\vec{\nabla} \times \vec{E} \right)}_{\vec{\nabla} \left(\vec{\nabla} \cdot \vec{E} \right) - \Delta \vec{E}} = - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}$$

$$\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0 \qquad (3)$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t} , \vec{\nabla} \right) \qquad (4)$$

$$\partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t} , \vec{\nabla} \right)$$

$$\left\{ \begin{array}{l} \partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} \\ x^\mu = (ct, \vec{r}) \\ x_\mu = (-ct, \vec{r}) \end{array} \right. \qquad (5)$$

$$\partial^2 \vec{E} = \partial_\mu \partial^\mu \vec{E} = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta \right) \vec{E} \underset{(3)}{=} 0 \qquad (6)$$

$$\partial^2 E = \partial_\mu \partial^\mu E = \frac{\partial^2}{\partial x^\mu \partial x_\mu} E = \qquad (7)$$

$$= -\frac{1}{c^2} \ddot{\vec{E}} + \Delta \vec{E} = 0$$

$$\begin{cases} \partial^2 \vec{B} = -\frac{1}{c^2} \ddot{\vec{B}} + \Delta \vec{B} = 0 \\ \partial^2 \vec{E} = -\frac{1}{c^2} \ddot{\vec{E}} + \Delta \vec{E} = 0 \end{cases} \qquad (8)$$

<u>One dimensional waves</u> (1+1)

$$\vec{E}(\vec{r}) \Rightarrow \vec{E}(z) \qquad (9)$$

$$-\frac{1}{c^2} \ddot{\vec{E}} + \frac{\partial^2 \vec{E}}{\partial x^2} = 0 \qquad (10)$$

retardation

$$\vec{E}(x, t) = \vec{E}(x \mp ct) \qquad (11)$$

(11) $\Rightarrow$ Wave travels with the velocity c (12)

Monochromatic wave

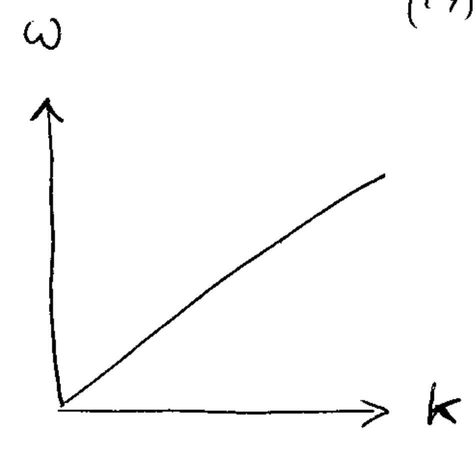
$$\vec{E}(\vec{r},t) = \vec{E}_0 \, e^{i(\vec{k}\vec{r}-\omega t)} \qquad (13)$$

{ Re or Im is suppressed here }

(8, 13) $\Rightarrow$

$$\vec{k}^{\,2} = \frac{\omega^2}{c^2} \qquad (14)$$

$$\omega \underset{(14)}{=} c k \qquad (15)$$



(16)

$$(15) \Rightarrow \begin{cases} c = \dfrac{\omega}{k} & (17) \\ \text{phase velocity} & \\ c = \dfrac{d\omega}{dk} & (18) \\ \text{group velocity} & \end{cases}$$

$$(13) \Rightarrow \begin{cases} \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\vec{r} - \omega t)} \\ \vec{B} = \vec{B}_0 \, e^{i(\vec{k}\vec{r} - \omega t)} \end{cases} \quad (19)$$

Similar

$$(2, 19) \Rightarrow \quad \vec{k} \cdot \vec{E}_0 = \vec{k} \cdot \vec{B}_0 = 0 \qquad (20)$$

$$\begin{cases} \vec{k} \times \vec{E}_0 = \omega \vec{B}_0 & (21) \\ \vec{k} \times \vec{B}_0 = - \dfrac{\omega}{c^2} \vec{E}_0 & (22) \end{cases}$$

Verify consistency (21, 22) :

$$\underbrace{\vec{k} \times (\vec{k} \times \vec{E}_0)}_{} \underset{(21)}{=} \omega \; \vec{k} \times \vec{B}_0 \qquad (23)$$

$$\underbrace{\vec{k}\,(\bar{k}\cdot\bar{E})}_{(20)} - k^2 \vec{E}$$

$$0$$

$$\Downarrow$$

$$(23) \Rightarrow \quad \vec{k} \times \vec{B}_0 = - \frac{k^2}{\omega} \vec{E} = \qquad (24)$$

$$\underset{(14)}{=} - \frac{\omega}{c^2} \vec{E}_0$$

comp. (22)

i.e. (21) and (22)

is one and the same

Thus

$$\begin{cases} \vec{E} = \vec{E}_0 \, e^{i(\vec{k}\vec{r} - \omega t)} \\ \vec{B} = \vec{B}_0 \, e^{i(\vec{k}\vec{r} - \omega t)} \end{cases}$$

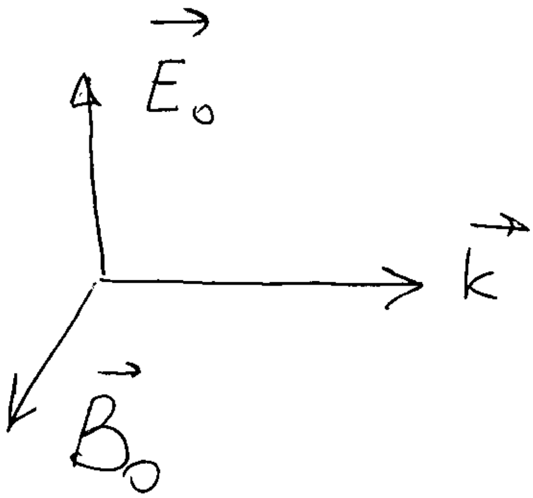
$$\omega = ck \qquad (25)$$

$$\underset{(20)}{\vec{E}_0 \perp \vec{k}} \qquad \underset{(20)}{\vec{B}_0 \perp k}$$

$$\underset{(21)}{\vec{E}_0 \perp \vec{B}_0} \qquad \vec{B}_0 = \frac{1}{c} \vec{n} \times \vec{E}_0$$

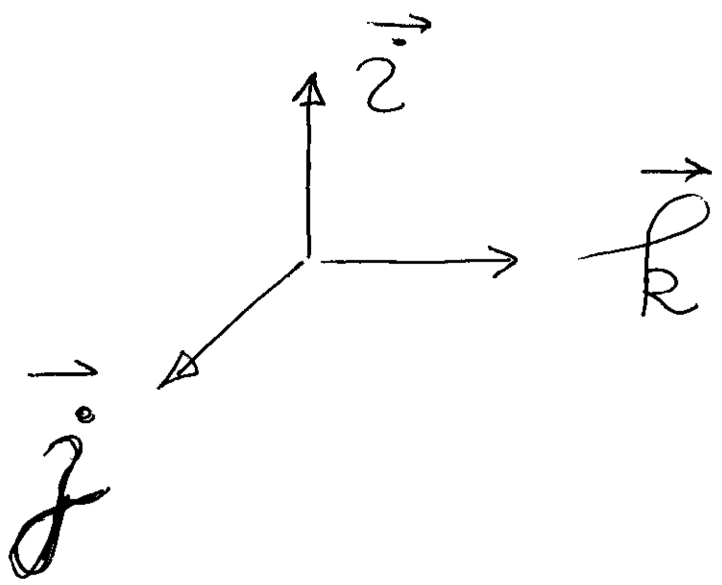
Polarization

$$\text{Take} \quad \left.\begin{aligned} \vec{E}_0 &- \text{real} \\ \vec{B}_0 &- \text{real} \end{aligned}\right\} \qquad (26)$$



linear polarization (27)

Circular polarization



$$\vec{i}^{\,2} = \vec{j}^{\,2} = \vec{k}^{\,2} = 1 \qquad (28)$$

$i^2 = -1$

$$\left\{\begin{array}{l} \text{Take} \quad \vec{k} = |\vec{k}|\,\vec{k} \\ \vec{E}_0 = E_0 \dfrac{\vec{i} + i\,\vec{j}}{\sqrt{2}} \\ \vec{B}_0 = B_0 \dfrac{\vec{j} - i\,\vec{i}}{\sqrt{2}} \end{array}\right. \qquad (29)$$

Circular polarization right (30)

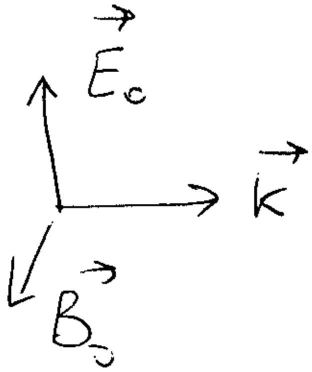
$i \to -i$ left

Energy, momentum

$$\vec{E} = \vec{E}_0 \cos\left(\overbrace{\vec{k}\vec{r} - \omega t}^{kz-\omega t}\right)$$

$$\vec{B} \overset{(13)}{=} \vec{B}_0 \cos\left(\vec{k}\vec{r} - \omega t\right) \quad \text{linear pol} \quad (31)$$

Recover Re



$$|\vec{E}_0| \underset{(21)}{=} c\,|\vec{B}_0| \qquad (32)$$

$$u_{em} = \frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0} = \qquad (33)$$

$$= \frac{1}{2} E_0^2 \cos^2\left(\underbrace{\vec{k}\vec{r} - \omega t}_{kz-\omega t}\right) \underbrace{\left[\varepsilon_0 + \frac{\overbrace{c^2}^{\varepsilon_0}}{\mu_0}\right]}_{2\varepsilon_0} =$$

$$= \varepsilon_0 \vec{E}_0^{\,2} \cos^2\left(\underbrace{\vec{k}\vec{r} - \omega t}_{kz-\omega t}\right) \qquad (34)$$

$$\langle a \rangle = \frac{1}{t}\int_0^t a\,dt\,\Big|_{t\to\infty} \qquad (35)$$

$$\langle u_{em} \rangle = \frac{\varepsilon_0 \vec{E}_0^{\,2}}{2} = \frac{\vec{B}_0^{\,2}}{2\mu_0} \qquad (36)$$

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} =$$

$$= \frac{\vec{E}_0\times\vec{B}_0}{\mu_0}\cos^2(kz-\omega t)$$

$$= \underbrace{\frac{E_0 B_0}{\mu_0}}\ \vec{k}\,\cos^2(\cancel{kz}-\omega t) \Rightarrow$$

$$c\,\frac{\vec{B}_0^{\,2}}{\mu_0} = c\,\varepsilon_0 \vec{E}_0^{\,2}$$

$$\Rightarrow \vec{S} = c\,\varepsilon_0 E_0^2 \cos^2(kz-\omega t)\,\vec{k} \qquad (37)$$

$$\langle \vec{S} \rangle = c \, \frac{\varepsilon_0 \vec{E}_0^{\,2}}{2} \, \vec{k} \qquad (38)$$

$$= c \, \frac{\vec{B}_0^{\,2}}{2\mu_0} \, \vec{k}$$

Comp. (36)

$$\langle \vec{P}_{em} \rangle = \frac{1}{c^2} \langle \vec{S} \rangle = \qquad (39)$$

$$= \frac{1}{c} \, \frac{\varepsilon_0 \vec{E}_0^{\,2}}{2} \, \vec{k}$$

$$c \, \| \vec{P}_{em} \| = u_{em} \, \vec{k}$$