VIII. Potentials and fields in the vacuum, wave equation

In the vacuum, where $\rho(\mathbf{r},t)=\mathbf{j}(\mathbf{r},t)=0$, Eqs.**Error! Reference source not found.** read

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)V(\mathbf{r},t)=0$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\mathbf{A}(\mathbf{r},t)=0 \qquad \textbf{(1.1)}$$

Comment. In the 4D notation the wave equations for the potentials read

$$\partial^2 A^\mu = 0 \qquad \textbf{(1.2)}$$

1. General properties of the wave equation, monochromatic solution

Generally, the equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)f(\mathbf{r},t)=0 \qquad \textbf{(1.3)}$$

where *c* is a constant is called the *wave equation*. A solution of Eq.(1.3) can be written

$$f(\mathbf{r},t)=\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix} f_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] \qquad \textbf{(1.4)}$$

where

$$\omega = ck \qquad \textbf{(1.5)}$$

Eq. (1.4) describes a monochromatic solution (which is also called a plane-wave), which according to Eq.(1.5) propagates with the velocity *c*. In the case at hand

$$c=\frac{\omega}{k}=\frac{d\omega}{dk} \qquad \textbf{(1.6)}$$

A linear combination of monochromatic solutions is again a solution of the wave equation. Therefore

$$f(\mathbf{r},t)=\int \tilde{f}(\mathbf{k})\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]\frac{d^3k}{\left(2\pi\right)^3} \qquad \textbf{(1.7)}$$

where $\tilde{f}(\mathbf{k})$ are arbitrary coefficients, is a solution. At the initial moment of time, say at $t=0$, Eq.(1.7) gives

$$f(\mathbf{r},0)=\int \tilde{f}(\mathbf{k})\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)\frac{d^3k}{\left(2\pi\right)^3} \qquad \textbf{(1.8)}$$

This means that $\tilde{f}(\mathbf{k})$ are the Fourier coefficients of $f(\mathbf{r},t)$, which can be found from

$$\tilde{f}(\mathbf{k}) = \int f(\mathbf{r},0)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)\frac{d^3k}{\left(2\pi\right)^3} \qquad \textbf{(1.9)}$$

Eqs.(1.7),(1.9) define the solution of the wave equation $f(\mathbf{r},t)$ for all *t*, if $f(\mathbf{r},0)$ is known.

2. Propagation of the wave packet

Consider slightly more general situation, when connection between ω and $\mathbf{k}$ is not necessarily linear, as in Eq.(1.6). Presume that $\omega_{\mathbf{k}}$ is some function of $\mathbf{k}$. In this case consider

$$f(\mathbf{r},t) = \int \tilde{f}(\mathbf{k})\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}\,t\right)\right]\frac{d^3k}{\left(2\pi\right)^3} \qquad \textbf{(1.10)}$$

at large distances at large momenta of time. Then, as a function of $\mathbf{k}$ the argument oscillates as crazy. Correspondingly, the integral is small. The only exception from this ruling gives a point (points) where the derivative of the phase turns zero

$$0 = \frac{\partial}{\partial\mathbf{k}}\left(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}\,t\right) = \mathbf{r}-\frac{\partial\omega_{\mathbf{k}}}{\partial\mathbf{k}}t \qquad \textbf{(1.11)}$$

In the vicinity of this point the variation of the phase is small. Therefore this point of integration gives a large contribution to the integral. As a result, the value of $f(\mathbf{r},t)$ is related to those momenta, which satisfy Eq.(1.11). From this equation one finds

$$\mathbf{v} = \frac{r}{t} = \frac{\partial\omega_{\mathbf{k}}}{\partial\mathbf{k}} \qquad \textbf{(1.12)}$$

The right-hand side here is called the *group* velocity. For linear electrodynamics (when $\omega = ck$) the group velocity coincides with *c*.

3. Spherical solutions

An example of a *spherical* monochromatic wave presents the function

$$f(\mathbf{r},t) = \frac{1}{r}\exp\left(i\,k\,r\right) \qquad \textbf{(1.13)}$$

This is a spherical solution, the so called s-wave (there exist other, more sophisticated spherical waves).

> **Exercise.** Verify that Eq.(1.13) gives a solution for the wave equation.
> Hint. Remember that the Laplacian in the spherical coordinates reads
>
> $$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\right] \qquad \textbf{(1.14)}$$
>
> The function $f(\mathbf{r},t)$ in (1.13) depends only on the radius. Therefore for this function
>
> $$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r}\frac{d^2}{dr^2}\left[\exp\left(i\,k\,r\right)\right] = -k^2 f \qquad \textbf{(1.15)}$$

4. 1+1 D

In 1+1 D case the wave equation (1.3) simplifies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)f(z,t)=0 \qquad \textbf{(1.16)}$$

Clearly, any function

$$f(z,t)=\phi_-(z-ct)+\phi_+(z+ct) \qquad \textbf{(1.17)}$$

satisfies the wave equation. The $\phi_-(z-ct)$ is usually called a *retarded* solution. Clearly, c describes a velocity of the propagation of the wave.

5. Wave equation for the electromagnetic fields

Differentiating the wave equations (1.1) and using definitions **Error! Reference source not found.** one concludes that the fields in the vacuum also satisfy the wave equations

$$\begin{aligned}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\mathbf{E}(\mathbf{r},t)=0\\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\mathbf{B}(\mathbf{r},t)=0\end{aligned} \qquad \textbf{(1.18)}$$

Exercise. Derive the wave equations for the fields in the vacuum directly from the Maxwell's equations

Hint: take rotor in the Faraday's law and use the Ampere's law (or vise versa, take rotor in the Ampere's law and use the Faraday's law). Remember that $\nabla\times\left(\nabla\times\mathbf{C}\right)=\nabla\left(\nabla\cdot\mathbf{C}\right)-\Delta\,\mathbf{C}$. Apply the Gauss's law to eliminate $\nabla\cdot\mathbf{E}$ and/or $\nabla\cdot\mathbf{B}$.

Assume that the solution of the wave equations is written in the form of Eq.(1.4), i. e.

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] \qquad \textbf{(1.19)}$$

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right] \qquad \textbf{(1.20)}$$

To simplify notation the sign Re (or Im) is omitted here.
The Gauss laws (electric + magnetic) give

$$\mathbf{k}\cdot\mathbf{E}=\mathbf{k}\cdot\mathbf{B}=0 \qquad \textbf{(1.21)}$$

The Faraday's law gives

$$\mathbf{k}\times\mathbf{E}_0=\omega\mathbf{B}_0 \qquad \textbf{(1.22)}$$

The Ampere's law ensures that

$$\mathbf{k}\times\mathbf{B}_0=-\frac{\omega}{c^2}\mathbf{E}_0 \qquad \textbf{(1.23)}$$

Eq.(1.5) allows one to simplify these equations

$$\begin{aligned} &\mathbf{n}\cdot\mathbf{E}=0, \quad \mathbf{n}=\mathbf{k}/k \\ &\mathbf{n}\cdot\mathbf{B}=0, \\ &\mathbf{n}\times\mathbf{E}_0=c\,\mathbf{B}_0, \\ &\mathbf{n}\times\mathbf{B}_0=-\frac{1}{c}\mathbf{E}_0 \end{aligned} \tag{1.24}$$

- **Linear polarization**

$$\begin{aligned} &\mathbf{n}=(0,0,1) \\ &\mathbf{E}_0=E_0\,(1,0,0) \\ &\mathbf{B}_0=B_0\,(0,1,0) \\ &E_0=cB_0 \end{aligned} \tag{1.25}$$

The wave propagates along the z-axes, the electric and magnetic fields are polarized along the x and y axes respectively.

$$\begin{aligned} \mathbf{E}(\mathbf{r},t)&=\mathrm{Re}\left\{E_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]\right\}(1,0,0)= \\ &|E_0|\cos\left(k\,z-\omega t+\phi\right)(1,0,0) \\ \mathbf{B}(\mathbf{r},t)&=\mathrm{Re}\left\{B_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right]\right\}(0,1,0)= \\ &|B_0|\cos\left(k\,z-\omega t+\phi\right)(0,1,0) \end{aligned} \tag{1.26}$$

This means that

$$\begin{aligned} &\mathbf{E}(\mathbf{r},t)=(E_x,0,0) \\ &E_x=|E_0|\cos\left(k\,z-\omega t+\phi\right) \\ &\mathbf{B}(\mathbf{r},t)=(0,B_y,0) \\ &B_y=|B_0|\cos\left(k\,z-\omega t+\phi\right) \end{aligned} \tag{1.27}$$

Eqs.(1.27) show that the fields are oscillating in space and time.

- **Circular polarization**

$$\begin{aligned} &\mathbf{n}=(0,0,1) \\ &\mathbf{E}_0=E_0\,(1,\,i,\,0) \\ &\mathbf{B}_0=B_0(-i,1,\,0) \\ &E_0=cB_0 \end{aligned} \tag{1.28}$$

This means that

$$\begin{aligned} &\mathbf{E}(\mathbf{r},t)=\mathrm{Re}\left\{E_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right](1,i,0)\right\} \\ &=|E_0|\left\{\cos\left(k\,z-\omega t+\phi\right)(1,0,0)-\sin\left(k\,z-\omega t+\phi\right)(0,1,0)\right\} \\ &=|E_0|\left(\cos\left(k\,z-\omega t+\phi\right),-\sin\left(k\,z-\omega t+\phi\right),0\right) \\ &\mathbf{B}(\mathbf{r},t)=\mathrm{Re}\left\{B_0\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)\right](-i,\,1,0)\right\} \\ &=|B_0|\left\{\sin\left(k\,z-\omega t+\phi\right)(1,0,0)+\cos\left(k\,z-\omega t+\phi\right)(0,1,0)\right\} \\ &=|B_0|\left(\sin\left(k\,z-\omega t+\phi\right),\,\cos\left(k\,z-\omega t+\phi\right),0\right) \end{aligned} \tag{1.29}$$

Eqs.(1.29) show that both fields are rotating in *x*-*y* plane with variation of *z* and *t*. For a fixed coordinate *z* the fields **E**, **B** are rotating *clockwise* on the *x*-*y* plane when time increases. Correspondingly, for a fixes *t* the fields **E**, **B** exhibit the *left*-hand spiral structure as functions of z . These properties correspond to the *right*-hand polarization; the *left*-hand polarization is described similarly, with a substitution $i \to -i$ made in Eq. (1.28). is made.