Quantization of monopole charge

Charge inside solenoid

(All calculations in CGS)

Orbital momentum of the EM field is

$$L_z \underset{(16)}{=} -\frac{\alpha}{2\pi} \frac{\phi}{e} \hbar$$

The momentum is quantized

$$L_z = \frac{\hbar}{2} \cdot n$$

This makes 1) the flux quantized

$$\phi \underset{(15)}{=} \frac{2\pi}{\alpha} e \frac{n}{2}$$

and 2) the monopole charge quantized

$$e_m \underset{(22)}{=} \frac{2\pi e}{\alpha} \frac{n}{2}$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{4\pi} \quad (CGS) \qquad (1)$$

$$\vec{L} = \frac{1}{c}\int \vec{r} \times \vec{S}\, d^3r \qquad (2)$$

$$= \frac{1}{4\pi c}\int \vec{r} \times \vec{S}\, d^3r$$

$$\vec{E} = \frac{e\vec{r}}{r^3} \leftarrow CGS \qquad (3)$$

$$\vec{L} = \frac{e}{4\pi c}\int \frac{\vec{r} \times (\vec{r} \times \vec{B})}{r^3}\, d^3r \qquad (4)$$

$$\vec{L} = \frac{e}{4\pi c} \int \frac{\vec{r}(\vec{r}\cdot\vec{B}) - r^2 \vec{B}}{r^3}\, d^3 r \qquad (5)$$

$$\vec{B} = (0, 0, B(\rho)) \qquad (6)$$

$$\vec{L} \underset{(5,6)}{=} L_z \qquad (7)$$

$$L_z \underset{(5,7)}{=} \frac{e}{4\pi c} \int \frac{z^2 - r^2}{r^3} B \,\rho\, d\rho\, d\varphi\, dz$$

$$= -\frac{e}{4\pi c} \cdot \int \frac{B(\rho)\rho^2}{r^3} \rho\, d\rho\, d\varphi\, dz \qquad (8)$$

$$\int_{-\infty}^{\infty} \frac{dz}{(\rho^2+z^2)^{3/2}} = \frac{d}{\rho^2} \underset{(12)}{=} \frac{2}{\rho^2} \qquad (9)$$

$$d = \int_{-\infty}^{\infty} \frac{dz}{(z^2+1)^{3/2}} \tag{10}$$

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^{\infty} dt\, t^{n-1} e^{-xt} \tag{11}$$

$$d \underset{(10,11)}{=} \frac{1}{\Gamma(3/2)} \int_0^{\infty} dt\, t^{3/2-1} \underbrace{\int_{-\infty}^{\infty} e^{-t(z^2+1)}\, dz}_{\sqrt{\frac{\pi}{t}}\, e^{-t}}$$

$$= \frac{\sqrt{\pi}}{\Gamma(3/2)} \underbrace{\int_0^{\infty} dt\, \sqrt{t}\, \frac{1}{\sqrt{t}}\, e^{-t}}_{1} = \tag{12}$$

$$\underset{(13)}{=} 2$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(1+\tfrac{1}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \tag{13}$$

$$L_z \underset{(8,9)}{=} -\frac{e}{4\pi c}\cdot 2\int \frac{B\rho^2}{\rho_1^2}\,\rho\, d\rho\, d\varphi$$

$$= -\frac{e}{2\pi c}\,\Phi \qquad (14)$$

$$\Phi = \int \vec{B}\, d\vec{S} \qquad (15)$$

$$L_z \underset{(14)}{=} -\frac{\alpha}{2\pi}\left(\frac{\Phi}{e}\right)\hbar \qquad (16)$$

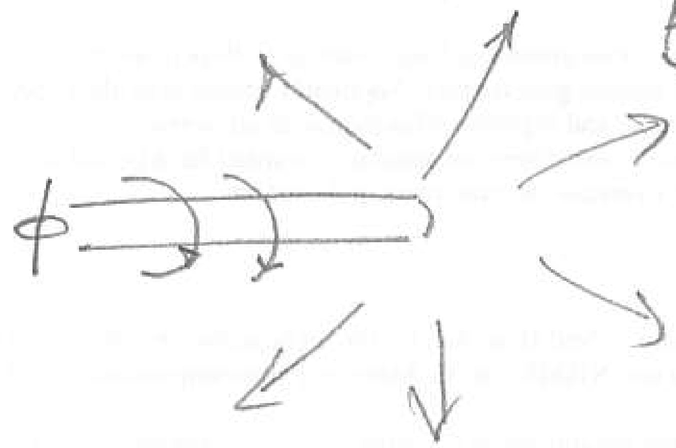
$$\left\{ \begin{array}{l} L_z = \frac{n}{2}\hbar \\ \boxed{\Phi \underset{(16)}{=} \frac{2\pi e}{\alpha}\cdot\frac{n}{2}} \end{array} \right. \qquad (17)$$

$n = 0, \pm 1, \ldots$

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036\ldots} \qquad (18)$$

CGS

$$\vec{B} = \frac{\phi \vec{r}}{r^2} \qquad (19)$$



$$\oint \vec{B}\, d\vec{s} \underset{(19)}{=} \phi \qquad (20)$$

$$\phi \underset{(19,20)}{\Rightarrow} e_m \text{ — magnetic charge} \qquad (21)$$

$$e_m \underset{(17,21)}{=} \frac{2\pi}{\alpha}\, e\, \frac{h}{2} \qquad (22)$$

Dirac monopole