Poynting's theorem

$$\vec{F} \cdot d\vec{\ell} = q\left(\vec{E} + \vec{v} \times B\right) \cdot \vec{v}\,dt$$

$$= \underbrace{q}_{\rho d^3 r} \vec{E} \cdot \vec{v}\,dt \qquad (1)$$

$$\vec{J} = \rho \vec{v}$$

$$\frac{dW}{dt} \underset{(1)}{=} \int \vec{E} \cdot \vec{J}\, d^3 r \qquad (2)$$

$$\vec{E} \cdot \vec{J} \underset{(4)}{=} \frac{1}{\mu_0} \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \qquad (3)$$

$$\nabla \times B = \underbrace{\frac{1}{c^2}}_{\varepsilon_0 \mu_0} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \qquad (4)$$

$$\nabla \cdot \left(\vec{E} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) \quad (5)$$

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) \underset{(5)}{=} - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \underbrace{\vec{B} \cdot \frac{\partial \vec{B}}{\partial t}}_{-\frac{1}{2}\frac{\partial}{\partial t}\vec{B}^2} \quad (6)$$

$$\vec{E} \cdot \vec{J} \underset{(3,6)}{=} - \frac{1}{\mu_0} \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) - \frac{d}{dt} \frac{1}{2} \left(\frac{\varepsilon_0}{2} \vec{E}^2 + \frac{1}{2\mu_0} \vec{B}^2\right) \quad (7)$$

$$\frac{dW}{dt} \underset{(2,7)}{=} - \frac{d}{dt} \int \left(\frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0}\right) d^3r - \frac{1}{\mu_0} \oint \vec{E} \times \vec{B} \cdot d\vec{S} \quad (8)$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \qquad (9)$$

$$\frac{dW}{dt} = - \frac{d\, W_{em}}{dt} - \oint_S \vec{S} \cdot d\vec{s}$$

$$\downarrow \qquad \downarrow$$

$$- \frac{d}{dt} \int^{v} \left(\frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0} \right)$$

$$\boxed{\frac{\partial}{\partial t} \left(\underbrace{\vec{E} \cdot \vec{J}}_{U_{mech}} + \underbrace{\frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0}}_{U_{em}} \right) = - \vec{\nabla} \cdot \vec{S}} \qquad (10)$$

$\vec{S}$ – energy flux

Alternative derivation

$$\vec{E}\quad\vec{B}\quad\begin{cases}\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J} & (11)\\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & (12)\end{cases}$$

$$\overbrace{\vec{E}\cdot(\nabla\times B) - B\,(\vec{\nabla}\times\vec{E})}^{-\nabla\cdot\underbrace{(\vec{E}\times\vec{B})}} \underset{(11,12)}{=}$$

$$= \frac{\partial}{\partial t}\left(\overbrace{\frac{1}{c^2}}^{\varepsilon_0\mu_0}\vec{E}\frac{\partial\vec{E}}{\partial t} + \vec{B}\frac{\partial\vec{B}}{\partial t}\right)$$

$$+\mu_0\,\vec{E}\cdot\vec{J} \qquad (13)$$

$$\frac{\partial}{\partial t}\,\frac{1}{2}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)\underset{(13)}{=} \qquad (14)$$

$$= -\vec{E}\cdot\vec{J} - \vec{\nabla}\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)$$

$$(16)$$

$$\frac{\partial}{\partial t}\int\left(\frac{\varepsilon_0\vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0}\right)d^3r \underset{(14)}{=} -\int\vec{E}\cdot\vec{J}\,d^3r$$

$$-\oint\vec{S}\cdot d\vec{s}$$

$$\int \vec{E}\cdot\vec{J}\, d^3r = \sum_i q\,\vec{v}\cdot\vec{E} = \frac{d\cancel{K}}{dt} \quad (17)$$

Kinetik energy

$$\frac{\partial}{\partial t}\left\{ \int' \left(\frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0} \right) d^3r + K \right\}$$

$$= -\oint \vec{S}' \cdot d\vec{s} \quad (18)$$

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0} + \underbrace{k}_{u_{mech}} \right) = -\vec{\nabla}\cdot\vec{S}' \quad (19)$$

$$K = \int \overbrace{u_{mech}}^{k}\, d^3r$$

Compare (15)