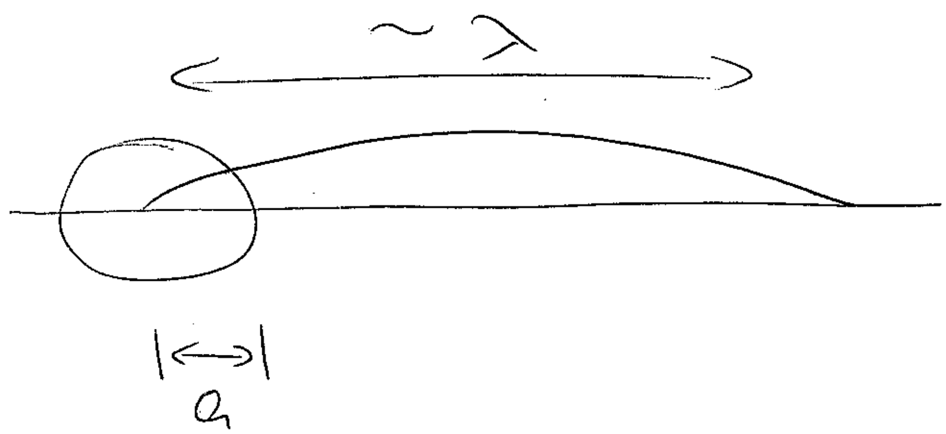
Radiation

Dipole radiation

$$a \ll \lambda \qquad (1)$$



(2)

$$\left.\begin{aligned} T &\cong \frac{a}{v} \\ \lambda &= cT \cong \frac{ca}{v} \end{aligned}\right\} \qquad (3)$$

$$\boxed{v \ll c} \qquad (4)$$

$$\begin{cases} d\mathcal{E} = \left(\frac{1}{\cancel{4\pi}}\right) \frac{2}{3c^3} \left|\ddot{\vec{d}}\right|^2 dt \\ \quad = \left(\frac{1}{\cancel{4\pi}}\right) \frac{2e^2}{3c^3} \vec{w}^{\,2} dt \quad (5) \qquad \vec{w} = \ddot{\vec{r}} \\ d\vec{P} = 0 \end{cases}$$

$$(4) \Rightarrow \begin{cases} v \simeq 0 & (6) \\ u^\mu \underset{(6)}{=} (1, 0\,0\,0) & \\ du^\mu = \left(0, \frac{1}{c}\frac{d\vec{v}}{dt}\right) dt = & \\ \qquad \overset{(8)}{=} (0, \vec{w}) \frac{ds}{c^2} & (7) \\ ds \overset{(6)}{=} c\,dt & (8) \end{cases}$$

$$\frac{du^\nu}{ds}\frac{du_\nu}{ds} \underset{(7)}{=} - \vec{w}^{\,2}\frac{1}{c^4} \qquad (9)$$

$$dP^\mu = \left(\frac{d\mathcal{E}}{c}, d\vec{P}\right) = \qquad (10)$$

$$\underset{(5,10)}{=} - \underbrace{\left(\frac{1}{4\pi}\right)\frac{2e^2}{3c}}_{\frac{e^2}{6\pi c}} \frac{du^\nu}{ds}\frac{du_\nu}{ds}\, dx^\mu$$

$$\Delta P^\mu = - \frac{e^2}{6\pi c}\int \frac{du^\nu}{ds}\frac{du_\nu}{ds}\, dx^\mu \qquad (11)$$

$$mc \frac{du^\mu}{ds} = \frac{e}{c} F^{\mu\nu} u_\nu \qquad (12)$$

$$\Delta P^\mu = -\frac{e^2}{6\pi c^5 m^2} \int \left(F^{\mu\lambda} u_\lambda\right)\left(F_{\mu\rho} u^\rho\right) dx^\mu \qquad (13)$$

$$\Delta \mathcal{E} = \frac{e^2}{6\pi c^3} \int_{-\infty}^{\infty} \frac{\vec{w}^{\,2} - \left(\frac{\vec{v} \times \vec{w}}{c}\right)^2}{\left(1 - \frac{v^2}{c^2}\right)^3} dt \qquad (14)$$

(11)

(19)

Derivation of (14)

$$u^\mu = \left(\frac{1}{\sqrt{1-v^2/c^2}} \; , \; \frac{\vec{v}/c}{\sqrt{1-v^2/c^2}} \right) \qquad (15)$$

$$du^\mu \underset{(15)}{=} \frac{\vec{v}\cdot d\vec{v}/c^2}{(1-v^2/c^2)^{3/2}} \left(1, \; \vec{v}/c \right)$$

$$+ \left(0 \; , \; \frac{d\vec{v}/c}{\sqrt{1-v^2/c^2}} \right) \qquad (16)$$

$$ds = c\, dt \sqrt{1-v^2/c^2} \qquad (17)$$

$$\frac{du^\mu}{ds} \underset{(16,17)}{=} \frac{\vec{v}\cdot\vec{w}}{c^3(1-v^2/c^2)^2} \left(1, \; \vec{v}/c \right)$$

$$+ \frac{1}{c^2(1-v^2/c^2)} \left(0, \; \vec{w} \right)$$

$$(18)$$

$$\left(\frac{du^\mu}{ds}\frac{du_\mu}{ds}\right)_{(18)} = \frac{1-v^2/c^2}{c^6(1-v^2/c^2)^4}\left(\vec{v}\cdot\vec{w}\right)^2$$

$$- \frac{\vec{w}^{\,2}}{c^4(1-v^2/c^2)^2}$$

$$- 2\,\frac{(\vec{v}\cdot\vec{w})^2}{c^{\cancel{6}}(1-v^2/c^2)^3} =$$

$$= - \frac{1}{c^4(1-v^2/c^2)^3}\left[\underbrace{\left(1-\frac{v^2}{c^2}\right)\vec{w}^{\,2} + \left(\frac{\vec{v}\cdot\vec{w}}{c}\right)^2}_{\vec{w}^{\,2} - \left(\frac{\vec{v}\times\vec{w}}{c}\right)^2}\right]$$

$$= - \frac{1}{c^4(1-v^2/c^2)^3}\left(\vec{w}^{\,2} - \left(\frac{\vec{v}\times\vec{w}}{c}\right)^2\right) \qquad (19)$$

End of derivation of (14)

$$m \frac{d}{dt} \frac{\vec{v}}{\sqrt{1-v^2/c^2}} = e\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) \quad (20)$$

$$\frac{\vec{W}}{\sqrt{1-v^2/c^2}} + \frac{\vec{v}\,(\vec{v}\cdot\vec{W})}{c^2\left(1-v^2/c^2\right)^{3/2}} \underset{(20)}{=} \frac{e}{m}\left(\bar{E} + \frac{\bar{v}\times\bar{B}}{c}\right) \quad (21)$$

$$\left(\vec{v}\cdot\vec{W}\right)\underbrace{\left(\frac{1}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2}{\left(1-v^2/c^2\right)^{3/2}}\right)}_{\frac{1-\cancel{v^2/c^2}+\cancel{v^2/c^2}}{\left(1-v^2/c^2\right)^{3/2}} = \frac{1}{\left(1-v^2/c^2\right)^{3/2}}} \underset{(21)}{=} \frac{e}{m}\,\vec{v}\cdot\vec{E}$$

$$\frac{\vec{v}\cdot\vec{W}}{\left(1-v^2/c^2\right)^{3/2}} = \frac{e}{m}\,\vec{v}\cdot\vec{E} \quad (22)$$

$(21, 22) \Rightarrow$

$$\boxed{\frac{\vec{W}}{\sqrt{1-v^2/c^2}} = \frac{e}{m}\left\{\vec{E} + \frac{\vec{v}\times\vec{B}}{c} - \frac{\vec{v}(\vec{v}\cdot\vec{E})}{c^2}\right\}} \quad (23)$$

$$\vec{W}^2 - \left(\frac{\vec{v}\times\vec{W}}{c}\right)^2 \equiv \left(1-\frac{v^2}{c^2}\right)\vec{W}^2 + \left(\frac{\vec{v}\cdot\vec{W}}{c}\right)^2 =$$

$$\underset{(22,23)}{=} \frac{e^2}{m^2}\left[\left(1-\frac{v^2}{c^2}\right)^2\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c} - \frac{\vec{v}(\vec{v}\cdot\vec{E})}{c^2}\right)^2 + \left(1-\frac{v^2}{c^2}\right)^3\frac{(\vec{v}\cdot\vec{E})^2}{c^2}\right] \Rightarrow \quad (24)$$

(24), (25)

$$\Rightarrow \frac{e^2}{m^2}\left(1-\frac{v^2}{c^2}\right)^2 \Bigg\{ \left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)^2 \underbrace{-2\left(\frac{\vec{v}\cdot\vec{E}}{c}\right)^2+\frac{v^2(\vec{v}\cdot\vec{E})^2}{c^4}+\left(1-\frac{v^2}{c^2}\right)\left(\frac{\vec{v}\cdot\vec{E}}{c}\right)^2}_{\left(\frac{\vec{v}\cdot\vec{E}}{c}\right)^2\left[\underbrace{-2+\cancel{\frac{v^2}{c^2}}+1-\cancel{\frac{v^2}{c^2}}}_{-1}\right]} \Bigg\}$$

$$\vec{W}^2-\left(\frac{\vec{V}\times\vec{W}}{c}\right)^2 \underset{(24,25)}{=} \frac{e^2}{m^2}\left(1-\frac{v^2}{c^2}\right)^2 \cdot \left\{\left(\vec{E}+\frac{\vec{V}\times\vec{B}}{c}\right)^2-\left(\frac{\vec{v}\cdot\vec{E}}{c}\right)^2\right\}$$

(26)

$$\Delta \mathcal{E} \underset{(14,26)}{=} \frac{e^4}{6\pi c^3 m^2} \int_{-\infty}^{\infty} \frac{\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)^2 - \left(\frac{\vec{v}\cdot\vec{E}}{c}\right)^2}{1 - v^2/c^2}\, dt \qquad (27)$$

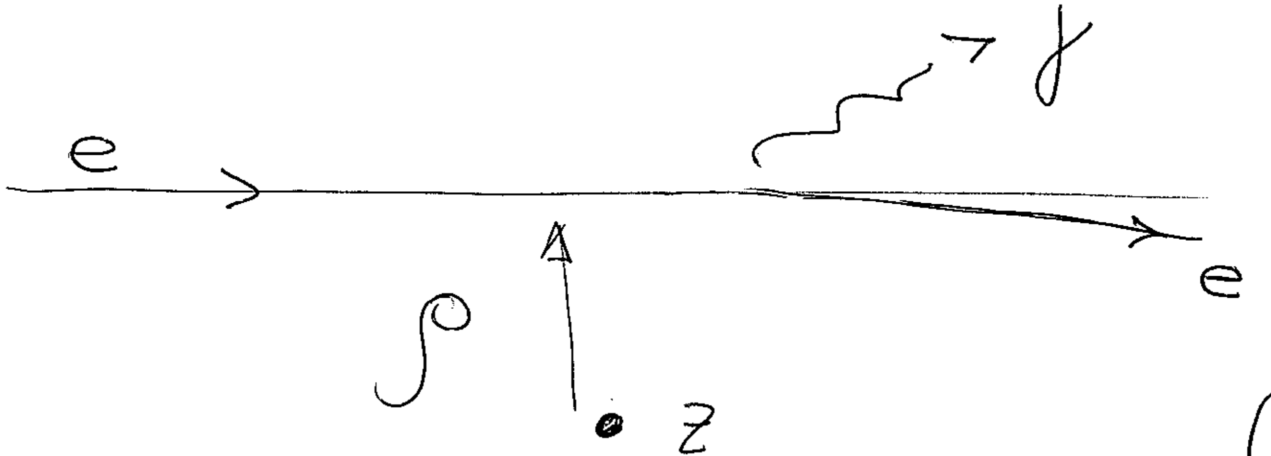
$$\Delta \mathcal{E} \underset{(27)}{\sim} \frac{1}{1 - v^2/c^2} \sim \gamma^2 \qquad (28)$$

$$\begin{cases} \text{Exception} \qquad B = 0 \qquad E \parallel V \\ \Delta \mathcal{E} \sim \gamma^0 \end{cases} \qquad (29)$$

$$(28) \Rightarrow \qquad \frac{\Delta \mathcal{E}}{\mathcal{E}} \sim \gamma \qquad (30)$$

Problem:

Radiation in ~~the~~ Coulomb field



(31)

$$\Delta \mathcal{E} = ?$$

Problem:

Find direction, in which there is no radiation.

$$\text{Eq. (33)} \Rightarrow \left(\vec{w} - \frac{\vec{\sigma}}{c}\right) \times \vec{W} = 0$$

Angular distribution

$$\vec{E} = \frac{e}{4\pi}\left\{\left(1-\frac{v^2}{c^2}\right)\frac{\vec{R}-\frac{\vec{v}}{c}R}{\tilde{R}^3} + \frac{\vec{R}\times\left(\left(\vec{R}-\frac{\vec{v}}{c}R\right)\times\vec{w}\right)}{c^2\,\tilde{R}^3}\right\} \tag{32}$$

$$\xrightarrow[R\to\infty]{} \frac{e}{4\pi R c^2}\cdot\frac{\vec{n}\times\left(\left(\vec{n}-\frac{\vec{v}}{c}\right)\times\vec{w}\right)}{\left(1-\frac{\vec{n}\cdot\vec{v}}{c}\right)^3} \tag{33}$$

$$1-\frac{\vec{n}\cdot\vec{v}}{c} = 1-\frac{v}{c}\cos\theta \cong$$

$$\underset{\theta\to 0}{\cong} 1-\frac{v}{c}\left(1-\frac{\theta^2}{2}\right) \underset{v\to c}{\simeq}$$

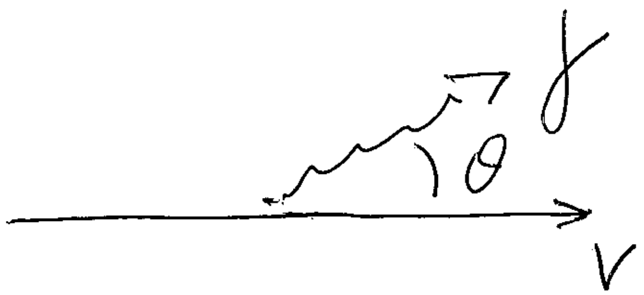
$$\cong 1-\frac{v}{c}+\frac{\theta^2}{2} \tag{34}$$

$$1 - \frac{\vec{n}\cdot\vec{v}}{c} \rightarrow 0$$

$$v \rightarrow c$$

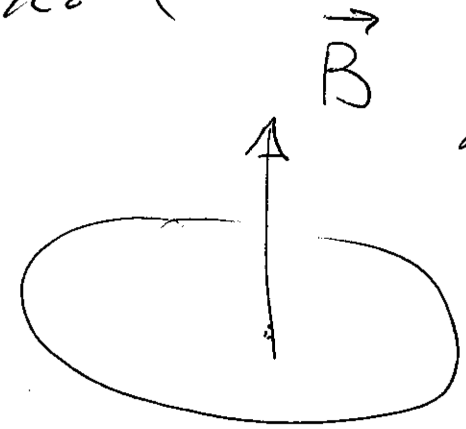
$$\vartheta \rightarrow 0$$

$$\Theta \underset{(34)}{\simeq} \sqrt{2\left(1-\frac{v}{c}\right)} = \sqrt{1-\frac{v^2}{c^2}} \quad (35)$$



Magnetic field

$$\frac{\vec{W}}{\sqrt{1-v^2/c^2}} \underset{(23)}{=} \frac{e}{m} \frac{\vec{v} \times \vec{B}}{c} \qquad (36)$$



$$\vec{n} = \frac{\vec{B}}{B} \qquad (37)$$

$$\vec{W} \underset{(37)}{=} \omega \cdot \vec{v} \times \vec{n} \qquad (38)$$

ω_c

$$\omega \underset{(36,37)}{=} \frac{eB}{mc} \sqrt{1-v^2/c^2} \qquad (39)$$

$$v = \omega r \qquad (40)$$

$$r \underset{(39,40)}{=} \frac{v}{\omega} = \frac{mc}{eB} \frac{v}{\sqrt{1-v^2/c^2}} \qquad (41)$$

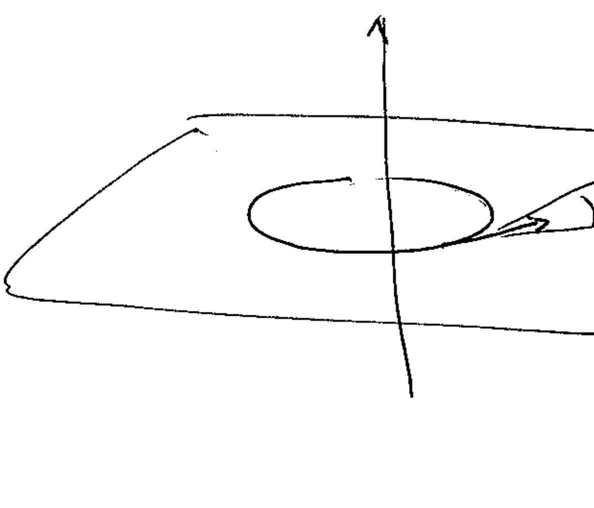
$$\frac{d\mathcal{E}}{dt} = \frac{e^4}{6\pi m^2 c^3} \frac{v^2 B^2}{c^2 \left(1 - \frac{v^2}{c^2}\right)} \tag{42}$$

$$\omega_c = \frac{eB}{mc} \tag{43}$$

$$\frac{d\mathcal{E}}{dt} = \frac{e^2}{6\pi c^3} \frac{\omega_c^2 v^2}{1 - v^2/c^2} =$$

$$= \frac{e^2}{6\pi c^3} \omega_c^2 v^2 \gamma^2 \tag{44}$$

<u>Angular distribution</u>



$$\theta \sim \sqrt{1 - \frac{v^2}{c^2}} \tag{45}$$

<u>Spectrum</u>

$$\omega \sim \omega_c \gamma^2 \tag{46}$$

Problem:

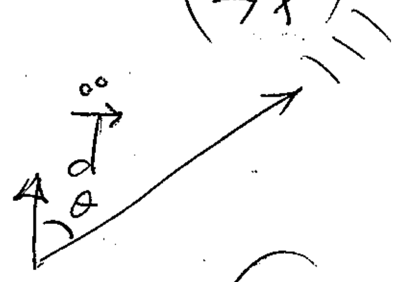
$$\frac{d\mathcal{E}}{dt} = f(t) \quad ?$$

Scattering by free charges

$$\bar{I} = \frac{d\mathcal{E}}{dt} \qquad (47)$$

$$\frac{dI}{d\Omega} = \frac{d^2\mathcal{E}}{dt\, d\Omega}$$

$$dI = \frac{\ddot{\vec{d}}^{\,2}}{4\pi c^3 m^2} \frac{\sin^2\theta\, d\Omega}{4\pi} \qquad (48)$$



$$\bar{I} = \frac{\ddot{\vec{d}}^{\,2}}{6\pi c^3 m^2} \qquad (49)$$

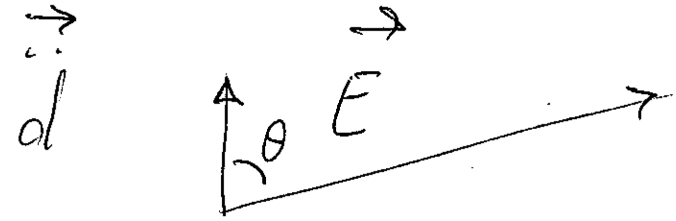
$$d\sigma = \frac{d\bar{I}}{S} \qquad (50)$$

$$\begin{cases} m\ddot{\vec{r}} = e\vec{E} \\ \ddot{\vec{d}} = \dfrac{e^2 \vec{E}}{m} \end{cases} \qquad (51)$$

$$dI \underset{(49,51)}{=} \frac{e^4 \vec{E}^2}{\cancel{4\pi} c^3 m^2} \frac{\sin^2\theta \, d\Omega}{\cancel{4\pi}} \qquad (52)$$

$$S = \frac{cE^2}{\cancel{8\pi}} \cdot \frac{\cancel{8\pi}}{1} = cE^2 \qquad (53)$$

$$d\sigma_{\Omega} = \frac{dI}{S} = \left(\frac{e^2}{4\pi mc^2}\right)^2 \sin^2\theta \, d\Omega \qquad (54)$$



$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{4\pi mc^2}\right)^2 \qquad (55)$$

$$r_0 = \frac{e^2}{4\pi m c^2}$$

$$m c^2 = \frac{e^2}{4\pi r_0}$$

$$\boxed{\sigma = \frac{8\pi}{3} r_0^2} \qquad (56)$$

Shortcut

$$\sigma = \frac{d\mathcal{E}}{dt}\,\frac{1}{S} \qquad (57)$$

$$\frac{d\mathcal{E}}{dt} = \frac{\ddot{\vec{d}}^{\,2}}{6\pi m^2 c^3} \qquad (58)$$

$$S = c\,E^2 \qquad (59)$$

$$\ddot{d} = \frac{e^2}{m} E \qquad (60)$$

$$\sigma = \frac{1}{6\pi}\,\frac{e^4}{m^2 c^4} = \frac{8\pi}{3}\left(\frac{e^2}{4\pi m c^2}\right)^2 \qquad (61)$$