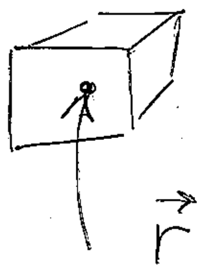
Retarding potentials

$$\partial^2 A^\mu = j^\mu \qquad (1)$$

$$\mu = 0$$

$$\partial^2 V(\vec{r}, t) = \rho(\vec{r}, t) \qquad (2)$$



$\vec{r}\,'$ (3)

$$\delta e(\vec{r}\,', t)$$

Consider a potential

δV created by $\delta e(\vec{r}\,', t)$

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) \delta V(\vec{R}, t) \underset{(2,3)}{=} \delta e(\vec{r}\,', t) \qquad (4)$$

$$\vec{R} = \vec{r} - \vec{r}\,' \qquad (5)$$

Almost everywhere (except $R \to 0$)

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) \delta V(\vec{R}, t) \underset{(4)}{=} 0 \qquad (6)$$

$$\delta V(\vec{R}, t) = \frac{\delta \chi(R, t)}{R} \qquad (7)$$

$$\Delta = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\right) + \cancel{\frac{\Lambda}{R^2}}$$

out

$$\Delta\left(\frac{\chi(R, t)}{R}\right) = \frac{1}{R}\frac{\partial^2 \chi}{\partial R^2} \qquad (8)$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial R^2}\right) \delta\chi(R,t) = 0 \quad (9)$$

$$\delta\chi(R,t) = \frac{\delta\chi(t-R)}{\cancel{R}} \quad (10)$$

retardation

$$\delta \bar{V}(\vec{R},t) \underset{(10,7)}{=} \frac{\delta\chi(t-R)}{R} \quad (11)$$

$$r \to r' \Rightarrow \vec{R} \to 0 \quad \text{(no retardation)} \quad (12)$$

$$\delta V(R\to 0, t) = \frac{\delta\chi(t)}{R} = \quad (13)$$

Coulomb law

$$= \frac{\delta e(\vec{r}\,', t)}{4\pi R}$$

$$\delta f(t) \underset{(13)}{=} \frac{1}{4\pi}\, \delta e(\vec{r}', t) \qquad (14)$$

$$\delta \bar{V}(\vec{R}, t) \underset{(11,14)}{=} \frac{\delta e(\vec{r}', t - R)}{4\pi R} \qquad (15)$$

$$R = |\bar{r} - \bar{r}'|$$

$$\bar{V}(\vec{r}, t) = \int \frac{\rho(\vec{r}', t - |\bar{r} - \bar{r}'|)}{4\pi |\vec{r} - \vec{r}'|}\, d^3r' \qquad (16)$$

$$V(\vec{r},t) \underset{(16)}{=} \int \frac{\bar{V}(\vec{r}\,',t')}{4\pi\,|\bar{r}-\bar{r}'|}\, d^3r' \qquad (17)$$

$$t' = t - |\vec{r}-\vec{r}\,'| \qquad (18)$$

$$A^{\mu}(\vec{r},t) = \int \frac{j^{\mu}(\vec{r}\,',t')}{4\pi\,|\bar{r}-\bar{r}'|}\, d^3r' \qquad (19)$$

Compare static case

$$V(\vec{r}) = \int \frac{\rho(\vec{r}\,')}{4\pi\,|\bar{r}-\bar{r}'|}\, d^3r' \qquad (20)$$

$$\left\{ \begin{aligned} V(\vec{r},t) &= \int \frac{\rho(\vec{r}',t')\, d^3r'}{4\pi\,|\vec{r}-\vec{r}'|} \\ \vec{A}(\vec{r},t) &= \frac{1}{c} \int \frac{\vec{j}(\vec{r}',t')}{4\pi\,|\vec{r}-\vec{r}'|}\, d^3r' \end{aligned} \right. \tag{21}$$

Miscellaneous Begin

$$V(\vec{r},t) \underset{(21)}{=} \int d^3r' \int d\tau \, \frac{\rho(\vec{r}',\tau)}{4\pi |\vec{r}-\vec{r}'|} \, \delta\left(\tau - t + |\vec{r}-\vec{r}'|\right) \quad (22)$$

$$\rho(\vec{r}',\tau) = e\,\delta\left(\vec{r}' - \vec{r}_0(\tau)\right) \quad (23)$$

$$V(\vec{r},t) = \int d\tau \, \frac{e}{4\pi |\vec{r}-\vec{r}_0(\tau)|} \cdot \delta\Big(\underbrace{\tau - t + |\vec{r}-\vec{r}_0(\tau)|}_{f(\tau)}\Big) \quad (24)$$

$$\delta\left(f(\tau)\right) = \frac{\delta(\tau - \tau_0)}{|f'(\tau_0)|} \quad (25)$$

$$f'(\tau) \underset{(25,27)}{=} 1 + \frac{\vec{v} \cdot \vec{R}}{R} \qquad (26)$$

$$\frac{d}{d\tau} \left| \vec{r} - \vec{r}_0(\tau) \right| = \frac{1}{2 \left| \vec{r} - \vec{r}_0(\tau) \right|} \cdot$$

$$\cdot \frac{d}{dr} \underbrace{\left(\vec{r} - \vec{r}_0(\tau) \right)^2}_{\vec{r}_0^{\,2}(\tau) - 2 \vec{r}_0(\tau) \cdot \vec{r} + r^2} =$$

$$= \frac{\vec{r}_0(\tau) \cdot \mathring{\vec{r}}_0(\tau) - \mathring{\vec{r}}_0(\tau) \cdot \vec{r}}{\left| \vec{r} - \vec{r}_0(\tau) \right|} =$$

$$= - \frac{\vec{V}_0(\tau) \cdot \left(\vec{r} - \vec{r}_0(\tau) \right)}{\left| \vec{r} - \vec{r}_0(\tau) \right|} =$$

$$= - \frac{\vec{V}_0(\tau) \cdot \vec{R}}{R} \qquad (27)$$

$$V(\vec{r},t) = \frac{e}{4\pi\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)} \tag{28}$$

$$\left\{\begin{aligned} \vec{R} &= \vec{r} - \vec{r}_0(t') \\ \vec{v} &= \vec{v}(t') \\ t - t' &= \frac{R(t')}{c} \end{aligned}\right. \tag{29}$$

$$\vec{A}(\vec{r},t) = \frac{e\vec{v}}{4\pi\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)} \tag{30}$$

Miscellaneous End

Short-cut

Take some moment of time t

For this t take a reference frame, in which $\vec{v}(t') = 0$

$$\begin{cases} V = \dfrac{e}{4\pi R} = \dfrac{e}{4\pi (t-t')} \\ \vec{A} = 0 \end{cases} \qquad (31)$$

$$A^\mu = \frac{e\, u^\mu}{4\pi\, R^\nu u_\nu} \qquad (32)$$

$t-t'$

$$\left.\begin{aligned} R^\mu &= (\ \ ,\ \vec{r} - \vec{r}\,') \\ u^\mu &= (1,\ \vec{0}) \end{aligned}\right. \qquad (33)$$

$$A^\mu = \frac{e\,u^\mu}{4\pi\,(R\,u)} \qquad (34)$$

$$u^\mu = \left(\frac{1}{\sqrt{1-v^2/c^2}} \,,\, \frac{\vec{v}}{c\sqrt{1-v^2/c^2}} \right) \qquad (35)$$

$$(R\,u) = \frac{c(t-t') - (\vec{r}-\vec{r}\,')\cdot\vec{v}/c}{\sqrt{1-v^2/c^2}} \qquad (36)$$

$$\left\{ \begin{aligned} V &= \frac{e}{4\pi\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)} \\ \vec{A} &= \frac{e\,\vec{v}}{4\pi c\left(R - \frac{\vec{v}\cdot\vec{R}}{c}\right)} \end{aligned} \right. \qquad (37)$$

Field

$$\vec{R}(t') = \vec{r} - \vec{r}_0(t') \qquad (1)$$

$$R(t') = t - t' \qquad (2)$$

$$R^2 = \vec{r}_0^{\,2} - 2\vec{r}\vec{r}_0 + \vec{r}^{\,2} \qquad (3)$$

$$= (t - t')^2$$

(3) $\Rightarrow$

$$2\,R\,\frac{\partial R}{\partial t} = -2(\vec r - \vec r_0)\vec v\,\frac{\partial t'}{\partial t}$$

$$= 2(t-t')\left(1-\frac{\partial t'}{\partial t}\right)$$

$$\Downarrow$$

$$R\frac{\partial R}{\partial t} = -(\vec R\cdot\vec v)\frac{\partial t'}{\partial t} = R\left(1-\frac{\partial t'}{\partial t}\right) \qquad (4)$$

$$\frac{\partial t'}{\partial t}\left(R - \vec R\cdot\vec v\right) \underset{(4)}{=} R$$

$$\Downarrow$$

$$\frac{\partial t'}{\partial t} = \frac{R}{\tilde R} \qquad (5)$$

$$\frac{\partial R}{\partial t} \underset{(4,5)}{=} -\frac{\vec R\cdot\vec v}{\tilde R} \qquad (6)$$

$$\tilde R = R - \vec v\cdot\vec R \qquad (7)$$

(3) $\Rightarrow$

$$\cancel{2} R \vec{\nabla} R = \cancel{2} \underbrace{(\vec{r} - \vec{r}_0)}_{\vec{R}} - \cancel{2} \vec{R} \cdot \vec{v} \underbrace{\frac{\partial t'}{\partial \vec{r}}}_{\vec{\nabla} t'}$$

$$= -\cancel{2} R \vec{\nabla} t' \qquad (8)$$

$$R \vec{\nabla} R = \vec{R} - (\vec{R} \cdot \vec{v}) \vec{\nabla} t' = -R \vec{\nabla} t' \qquad (9)$$

$$\vec{\nabla} t' (R - \vec{R} \cdot \vec{v}) = -\vec{R}$$

$$\left\{ \begin{array}{l} \vec{\nabla} t' \underset{(9)}{=} -\dfrac{\vec{R}}{\tilde{R}} \\ \vec{\nabla} R \underset{(9)}{=} \dfrac{\vec{R}}{\tilde{R}} \end{array} \right. \qquad (10)$$

Summary so far

$$\frac{\partial t'}{\partial t} \underset{(5)}{=} \frac{R}{\tilde{R}} \qquad (11)$$

$$\vec{\nabla} t' \underset{(10)}{=} - \frac{\vec{R}}{\tilde{R}}$$

$$\left\{ \begin{array}{l} \dfrac{\partial R}{\partial t} \underset{(6,10)}{=} - \dfrac{\vec{v} \cdot \vec{R}}{\tilde{R}} \\[2ex] \vec{\nabla} R = \dfrac{\vec{R}}{\tilde{R}} \end{array} \right. \qquad (12)$$

$$\frac{\partial \vec{R}}{\partial t} \underset{(1)}{=} - \vec{v}\,\frac{\partial t'}{\partial t} \underset{(11)}{=} - \frac{R}{\tilde{R}}\,\vec{v} \qquad (13)$$

$$\frac{\partial R_i}{\partial r_j} \underset{(1)}{=} \delta_{ij} - v_i\,\frac{\partial t'}{\partial r_j} =$$

$$\underset{(11)}{=} \delta_{ij} + \frac{v_i R_j}{\tilde{R}} \qquad (14)$$

$$\frac{\partial \tilde{R}}{\partial t} \underset{(7)}{=} \frac{\partial}{\partial t}\left(R - \vec{v}\cdot\vec{R}\right) =$$

$$\underset{\substack{(12)\\(13)}}{=} - \frac{\vec{v}\cdot\vec{R}}{\tilde{R}} + \frac{v^2 R}{\tilde{R}} - \dot{\vec{v}}\cdot\vec{R}\,\frac{\partial t'}{\partial t}$$

$$\underset{(11)}{=} - \frac{\vec{v}\cdot\vec{R} - v^2 R}{\tilde{R}} - \left(\dot{\vec{v}}\cdot\vec{R}\right)\frac{R}{\tilde{R}} \qquad (15)$$

Verify (14)

$$R_i \frac{\partial R_i}{\partial r_j} \underset{(14)}{=} \frac{1}{2} \frac{\partial}{\partial r_j}\left(R^2\right) = R_j + \frac{(\vec{R}\vec{\sigma})}{\tilde{R}}$$

$$= R \nabla_j R$$

$$\vec{\nabla} R = \underbrace{\left(1 + \frac{\bar{R}\bar{\sigma}}{\tilde{R}}\right)}_{\frac{\cancel{R}}{\tilde{R}}} \frac{\vec{R}}{\cancel{R}} = \frac{\vec{R}}{\tilde{R}}$$

Comp. (12)

Ok (14)

$$\vec{\nabla}_j \tilde{R} = \vec{\nabla}_j \left(R - \vec{v} \cdot \vec{R} \right) =$$

$$\underset{(12)}{=} \frac{\vec{R}_j}{\tilde{R}} - \vec{v}_j \left(\delta_{ij} + \frac{v_i R_i}{\tilde{R}} \right)$$

(14)

$$- \left(\dot{\vec{v}} \cdot \vec{R} \right) \cdot \frac{\partial t'}{\partial r_j} =$$

$$\underset{(11)}{=} \frac{\vec{R}}{\tilde{R}} - \vec{v} - \frac{v^2 \vec{R}}{\tilde{R}} + \left(\dot{\vec{v}} \cdot \vec{R} \right) \frac{\vec{R}}{\tilde{R}}$$

(16)

$$= \left(1 - v^2\right) \frac{\vec{R}}{\tilde{R}} - \vec{v} + \left(\dot{\vec{v}} \cdot \vec{R} \right) \frac{\vec{R}}{\tilde{R}}$$

$$\left\{\begin{aligned} \frac{\partial \tilde{R}}{\partial t} &\underset{(15)}{=} - \frac{\vec{v}\cdot\vec{R} - v^2 R}{\tilde{R}} - \left(\dot{\vec{v}}\cdot\vec{R}\right)\frac{R}{\tilde{R}} \\ \vec{\nabla}\tilde{R} &\underset{(16)}{=} -\vec{v} + \left(1 - v^2\right)\frac{\vec{R}}{\tilde{R}} + \left(\dot{\vec{v}}\cdot\vec{R}\right)\frac{\vec{R}}{\tilde{R}} \end{aligned}\right. \qquad (17)$$

$$\left\{\begin{aligned} \frac{\partial \vec{v}}{\partial t} &\underset{(11)}{=} \dot{\vec{v}}\,\frac{R}{\tilde{R}} \\ \frac{\partial \vec{V_i}}{\partial r_j} &= \dot{\vec{V_i}}\,\frac{\partial t'}{\partial r_j} \underset{(12)}{=} - \frac{\dot{\vec{V_i}}\,\vec{R_j}}{\tilde{R}} \end{aligned}\right. \qquad (18)$$

$$\left\{\begin{array}{l} V = \dfrac{e}{4\pi} \dfrac{1}{\tilde{R}} \\ \vec{A} = \dfrac{e}{4\pi} \dfrac{\vec{v}}{\tilde{R}} \end{array}\right. \qquad (19)$$

$$\left\{\begin{array}{l} \vec{E} = - \nabla V - \dfrac{1}{c} \dot{\vec{A}} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{array}\right. \qquad (20)$$

$$\vec{E} \underset{(20)}{=} \frac{e}{4\pi} \left\{ \frac{1}{\tilde{R}^2} \left[\vec{\nabla} \tilde{R} + \frac{1}{c} \vec{V} \cdot \frac{\partial \tilde{R}}{\partial t} \right] - \frac{1}{c} \frac{\dot{\vec{V}}}{\tilde{R}} \frac{\partial t'}{\partial t} \right\} \qquad (21)$$

$$\vec{\nabla} \tilde{R} + \frac{1}{c} \vec{V} \frac{\partial}{\partial t} \tilde{R} \underset{(17)}{=} \qquad (22)$$

$$\underset{(17)}{=} - \vec{\sigma} + (1-\sigma^2) \frac{\vec{R}}{\tilde{R}} + \left(\dot{\vec{\sigma}} \cdot \vec{R} \right) \frac{\vec{R}}{\tilde{R}}$$

$$- \frac{\vec{V}}{c} \left(\frac{\vec{\sigma} \cdot \vec{R} - \sigma^2 R}{\tilde{R}} + \frac{\left(\dot{\vec{\sigma}} \cdot \vec{R} \right) R}{\tilde{R}} \right)$$

$$= (1-\sigma^2) \frac{\vec{R}}{\tilde{R}} - \vec{\sigma} \left[1 - \sigma^2 \frac{R}{\tilde{R}} + \frac{\vec{\sigma} \cdot \vec{R}}{\tilde{R}} \right]$$

$$+ \left(\dot{\vec{\sigma}} \cdot \vec{R} \right) \frac{\cancel{1}}{\tilde{R}} \left[\vec{R} - \frac{\vec{V}}{c} R \right]$$

$$\vec{\nabla}\tilde{R} + \frac{1}{c}\vec{v}\frac{\partial\tilde{R}}{\partial t} \overset{(22)}{=}$$

$$= \left(1 - \frac{v^2}{c^2}\right)\frac{\vec{R}}{\tilde{R}} - \vec{v}\,\frac{R}{\tilde{R}}\left(1 - v^2\right)$$

$$+ \left(\vec{R} - \frac{\vec{v}}{c}R\right)\frac{\dot{\vec{v}}\cdot\vec{R}}{\tilde{R}} =$$

$$= \frac{1 - v^2/c^2}{\tilde{R}}\left(\vec{R} - \vec{v}R\right)$$

$$+ \frac{\dot{\vec{v}}\cdot\vec{R}}{\tilde{R}}\left(\vec{R} - \vec{v}R\right) \qquad (23)$$

$$\vec{E} \underset{(21,23)}{=} \frac{e}{4\pi} \left\{ \left(1 - \frac{v^2}{c^2}\right) \frac{\vec{R} - \vec{v} R}{\tilde{R}^3} + \frac{(\dot{\vec{v}} \cdot \vec{R})(\vec{R} - \vec{v} R)}{\tilde{R}^3} - \frac{\dot{\vec{V}} R}{c \tilde{R}^2} \right\} =$$

$$= \frac{e}{4\pi \tilde{R}^3} \left\{ \left(1 - \frac{v^2}{c^2}\right)(\vec{R} - \vec{v} R) + (\dot{\vec{v}} \cdot \vec{R})(\vec{R} - \vec{v} R) - \dot{\vec{V}} \underbrace{R \tilde{R}} \right\} \quad (24)$$

$$R \left(R - \frac{\vec{V} \cdot \vec{R}}{c}\right) = R^2 - \frac{1}{c}(\vec{V} \cdot \vec{R}) R$$
$$= \vec{R} \cdot \left(\vec{R} - \frac{\vec{V}}{c} R\right)$$

$$\vec{E} \underset{(24)}{=} \frac{e}{4\pi \tilde{R}^3}\left\{\left(1-\frac{v^2}{c^2}\right)\left(\vec{R}-\vec{v}R\right) + \frac{1}{c^2}\underbrace{\vec{R}\times\left(\left(\vec{R}-\frac{\vec{v}}{c}R\right)\times\dot{\vec{v}}\right)}_{(25)}\right\}$$

$$\left(\vec{R}-\frac{\vec{v}}{c}R\right)\left(\dot{\vec{v}}\cdot\vec{R}\right) - R\tilde{R}\,\dot{\vec{v}}$$

$$\vec{B} = \frac{e}{4\pi} \vec{\nabla} \times \frac{\vec{v}}{\tilde{R}} =$$

$$= \frac{e}{4\pi} \left\{ - \frac{1}{\tilde{R}^{3}} \vec{\nabla} \tilde{R} \times \vec{v} \right.$$

$$\left. + \frac{1}{\tilde{R}} \vec{\nabla} t' \times \dot{\vec{v}} \right\} =$$

$$\overset{(17)}{=} \frac{e}{4\pi} \left\{ - \frac{1}{\tilde{R}^{3}} \Big[(1 - v^2) \frac{\vec{R} \times \vec{v}}{\tilde{R}} + \right.$$

$$+ (\dot{\vec{v}} \cdot \vec{R}) \frac{\vec{R} \times \vec{v}}{\tilde{R}}$$

$$\left. - \frac{\vec{R} \times \dot{\vec{v}}}{\tilde{R}^2} \right\} \qquad (26)$$

$$\vec{B} \underset{(26)}{=} \frac{e}{4\pi} \frac{\vec{R}}{\tilde{R}} \times$$

$$\left\{ -(1-v^2)\frac{\vec{v}}{\tilde{R}^2} - \frac{(\dot{\vec{v}}\cdot\vec{R})\vec{v}}{\tilde{R}^2} - \frac{\dot{\vec{v}}}{\tilde{R}} \right\}$$

$$= \frac{\vec{R}}{R} \times \frac{e}{4\pi} \Bigg\{$$

$$-(1-v^2)\frac{R}{\tilde{R}^3}\vec{v} - \frac{(\dot{\vec{v}}\cdot\vec{R})R\vec{v}}{\tilde{R}^3} - \frac{R\dot{\vec{v}}}{\tilde{R}^2} \Bigg\} \qquad (27)$$

(25), (27)

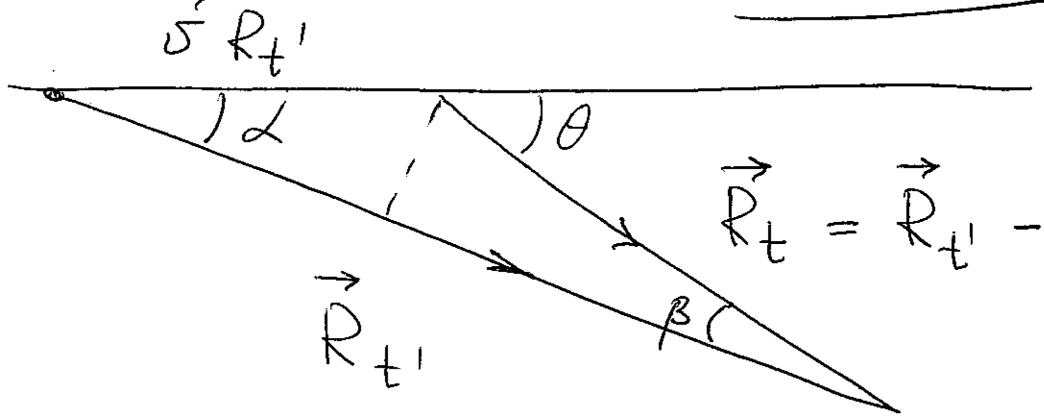
$$\vec{B} = \frac{\vec{R}\times\vec{E}}{R} \qquad (28)$$

$$\vec{E} = \frac{e}{4\pi}\left\{ (1-v^2)\,\frac{\vec{R} - \vec{v}R}{\tilde{R}^3} + \frac{\vec{R}\times\left[\left(\vec{R}-\vec{v}R\right)\times\dot{\vec{v}}\right]}{\tilde{R}^3} \right\} \qquad (29)$$

$$\vec{B} = \frac{\vec{R}\times\vec{E}}{R}$$

$$\tilde{R} = R - \vec{v}\cdot\vec{R}$$

$V = \text{Const}$



$$\vec{R}_t = \vec{R}_{t'} - \vec{v} R_{t'} \qquad (30)$$

$$R_{t'} \underset{(30)}{=} v R_{t'} \cos\alpha + R_t \cos\beta \qquad (31)$$

$$R_{t'} - \vec{v} \circ \vec{R}_{t'} = R_t \cos\beta = \qquad (32)$$

$$\underset{(33)}{=} R_t \sqrt{1 - v^2 \sin^2\theta}$$

$$\frac{\sin\beta}{v R_{t'}} = \frac{\sin\theta}{R_{t'}} \qquad \sin\beta = v \sin\theta \qquad (33)$$

$$\vec{E} = \frac{e \vec{R}_t}{4\pi R_t^3} \frac{1 - v^2/c^2}{\left(1 - (v^2/c^2)\sin^2\theta\right)^{3/2}} \qquad (34)$$

$\vec{V} = \text{Const}$

$$\frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$$