Equation of motion for potential

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (1)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (2)$$

$$\partial^\mu \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) = j_\nu \quad (3)$$

$$\partial^2 A_\nu - \partial_\nu \left(\partial^\mu A_\mu \right) = j_\nu \quad (4)$$

<u>Lorentz gauge</u>

$$\partial_\mu A^\mu = 0 \qquad (5)$$

$(4,5) \Rightarrow$

$$\boxed{\partial^2 A_\mu = j_\mu} \qquad (6)$$

$$\partial^2 = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta \qquad (7)$$

From the Coulomb law to the full-scale ED

Coulomb law $\quad E = \dfrac{e}{4\pi r^2} \qquad (8)$

$\Downarrow$

Gauss law $\quad \vec{\nabla}\cdot\vec{E} \underset{(10)}{=} \rho \qquad (9)$

$$\vec{E} \underset{(12)}{=} -\vec{\nabla} V \qquad (10)$$

$$\Downarrow$$

$$-\Delta V \underset{(11,12)}{=} \rho \qquad (11)$$

Lorentz inv $\Rightarrow$ $\qquad (12)$

$$\boxed{(13,15) \Longleftrightarrow (8)} \quad \left\{ \begin{array}{l} -\Delta \underset{(14)}{=} \dfrac{\partial^2}{\partial t^2} - \Delta \\ \rho \underset{14}{\rightarrow} j^\mu \\ V \underset{(14)}{\rightarrow} A_\mu \end{array} \right. \qquad (13)$$

Lagrangian, Action

Consider simple mechanical system, which has 1 coordinate, call it $Q = Q(t)$.

1) $$m\ddot{Q} = F = -\frac{\partial U}{\partial Q} \qquad (14)$$

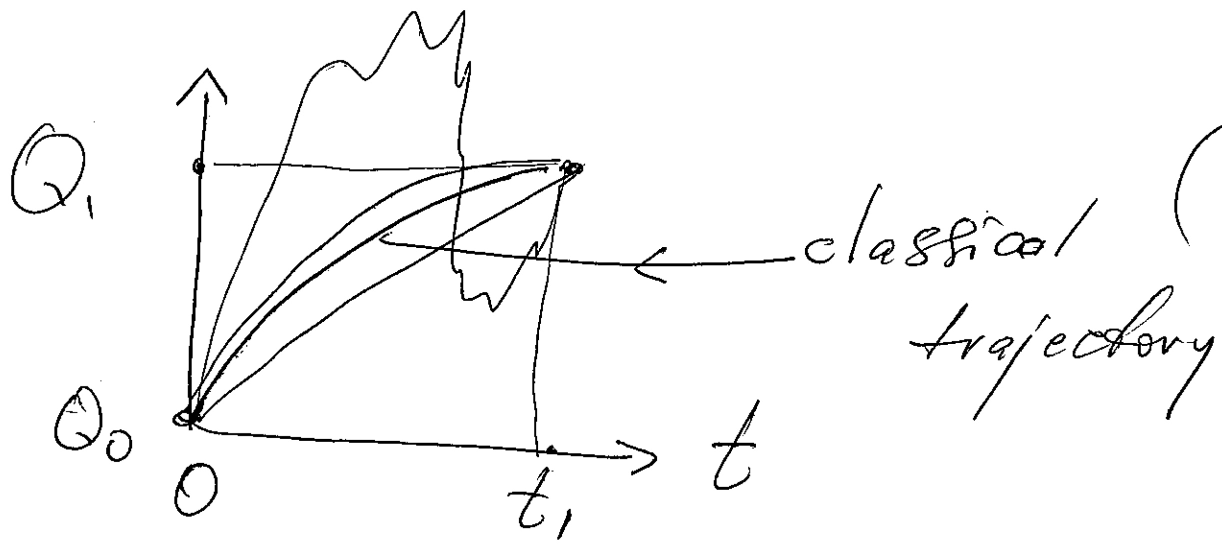
2. $$L = \frac{m\dot{Q}^2}{2} - U \qquad (15)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{\partial L}{\partial Q} \qquad (16)$$

Compare with energy $E = \frac{m\dot{Q}^2}{2} + U$

$$S\left[Q(t)\right] = \int_0^{t_1} L\left(Q(t'), \dot{Q}(t')\right) dt' \qquad (17)$$

$$\begin{cases} Q(t=0) = Q_0 \\ Q(t=t_1) = Q_1 \end{cases}$$



(18)

$$S\left[Q(t)\right] - S\left[Q_{cl}(t)\right] \equiv \delta S \qquad (19)$$

$$\delta S = 0 \qquad (20)$$

$$\delta S = \int_0^{t_1} \left[\frac{\partial L}{\partial Q}\, \delta Q(t) + \frac{\partial L}{\partial \dot Q}\, \delta \dot Q \right] dt$$

$$0$$

$$= \frac{\partial L}{\partial Q}\, \delta Q \Big|_0^{t} +$$

$$+ \int_0^{t_1} \delta Q \left[\frac{\partial L}{\partial Q} - \frac{d}{dt} \frac{\partial L}{\partial \dot Q} \right] dt \quad (21)$$

$$\delta S = 0 \underset{(20)}{\overset{(21)}{\Longleftrightarrow}} \frac{\partial L}{\partial Q} = \frac{d}{dt} \frac{\partial L}{\partial \dot Q}$$

(22)

Comp. (16)

Field theory

$$Q(t) \rightarrow Q(\vec{r}, t) \equiv Q(x) \qquad (23)$$

$$L \rightarrow \mathcal{L} = \mathcal{L}(Q, \underbrace{\partial_\mu Q}_{Q_{,\mu}}) \qquad (24)$$

$$S = \int \mathcal{L} \, d^4x \qquad (25)$$

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial Q} \delta Q + \frac{\partial \mathcal{L}}{\partial Q_{,\mu}} \delta Q_{,\mu} \right) d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial Q} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial Q_{,\mu}} \right) \right] \delta Q \, d^4x \qquad (26)$$

$$\delta S = 0 \tag{27}$$

$$\Big\Updownarrow$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (Q_{,\mu})} \underset{(27,26)}{=} \frac{\partial \mathcal{L}}{\partial Q} \tag{28}$$

$$\mathcal{L}_{em} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{\vec{E}^2 - \vec{B}^2}{2} \quad (29)$$

$$\left\{ \begin{array}{l} F_{0n} = E_n \\ F^{0n} = -E^n \\ F_{nm} = -\epsilon_{nm\ell} B_\ell \\ F^{nm} = -\epsilon_{nm\ell} B_\ell \end{array} \right. \quad (30)$$

from $F^{n0} F_{n0}$

$$2 \sum_n F^{0n} F_{0n} \underset{(30)}{=} -2 \vec{E}^2 \quad (31)$$

$$\sum_{n,m} F_{nm} F^{nm} \underset{(30)}{=} \sum_{\substack{nm \\ \ell k}} \underbrace{\epsilon_{nm\ell}\, \epsilon_{nmk}}_{2\delta_{\ell k}} B_\ell B_k =$$

$$= 2 \vec{B}^2 \quad (32)$$

$$\frac{d\mathcal{E}}{dV} = \frac{\vec{E}^2 + \vec{B}^2}{2} \qquad (33)$$

gauge cond $\quad V = 0 \qquad (34)$

$\vec{E} \underset{(34)}{=} -\dot{\vec{A}} \quad \longleftarrow$ "velocity"

$\vec{E}^2$ — "kinetic energy"

$\vec{B}^2$ — "potential energy"

	Enery	L
mechanical system	$\frac{m\dot{Q}^2}{2} + U$	$\frac{m\dot{Q}^2}{2} - U$
ED	$\frac{\vec{E}^2 + \vec{B}^2}{2}$	$\frac{\vec{E}^2 - \vec{B}^2}{2}$

$$\mathcal{L}_{em} \underset{(29)}{-} \text{Lorentz inv} \tag{35}$$

$$d^4x = d^3r\,dt \quad - \text{Lorentz inv} \tag{36}$$

$$S' \underset{(35,36)}{-} \text{Lorentz inv} \tag{37}$$

(as it should)

Compare for free particle

$$\begin{cases} \dfrac{S}{\hbar} = \vec{k}\cdot\vec{r} - \omega t \\ S' = \vec{p}\cdot\vec{r} - \mathcal{E}t = \\ \quad = -p^{\mu}x_{\mu} \end{cases}$$

$$\mathcal{L}_{int} = ?$$

Static Electric field

$$U = qV \tag{38}$$

$$\Downarrow$$

$$\delta L = -U = -\int V\rho\, d^3r \tag{39}$$

$$\Downarrow$$

$$\delta\mathcal{L} \underset{(39)}{=} -V\rho \tag{40}$$

make it Lorentz Inv (41)

$$\delta\mathcal{L}_{int} \underset{(40,41)}{=} -j_\mu A^\mu \tag{42}$$

$$S = \int \mathcal{L}\, d^4x \qquad (43)$$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A^{\mu} j_{\mu} \qquad (44)$$

$$\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu) \qquad (45)$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} \underset{(44)}{=} - j^{\nu} \qquad (46)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \underset{(44)}{=} -\frac{1}{4} \cdot 2 \cdot 2 \cdot F^{\mu\nu} = - F^{\mu\nu} \qquad (47)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \underset{(28)}{=} \frac{\partial \mathcal{L}}{\partial A_\nu} \qquad (48)$$

$$\partial_\mu F^{\mu\nu} \underset{(46,47,48)}{=} j^\nu \qquad 49$$

Short-cut to formulate ED

$$\frac{d\mathcal{E}}{dV} = \frac{\vec{E}^2 + B^2}{2} \Rightarrow \mathcal{L}_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$U = qV \Rightarrow \mathcal{L}_{int} = -j^{\mu} A_{\mu}$$

$$\boxed{\begin{aligned} S &= \int \mathcal{L} \, d^4x \\ \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^{\mu} A_{\mu} \end{aligned}}$$

(50)