Definition of Fields

$$\begin{cases} \vec{E} = -\vec{\nabla} V - \frac{1}{c}\dot{\vec{A}} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases} \qquad (1)$$

$$B_n = \epsilon_{nml} \nabla_m A_l \qquad (2)$$

$$n, m, l = 1, 2, 3$$

$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \end{cases} \qquad (3)$$

$$F_{\mu\nu} \underset{(3)}{=} - F_{\nu\mu} \qquad (4)$$

$$\begin{cases} F_{0n} \underset{(3)}{=} \vec{E}_n = - F_{n0} = - F^{0n} = F^{n0} \\ F_{nm} = - \epsilon_{nml} B_l = - F_{mn} = F^{nm} = - F^{mn} \end{cases} \qquad (5)$$

$$F_{\mu\nu} \underset{(3)}{=} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} \underset{(3)}{=} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (6)$$

$$F_{\mu\nu} = \left(\vec{E}, \vec{B}\right)$$

$$F^{\mu\nu} = \left(-\vec{E}, \vec{B}\right) \quad (7)$$

$$\tilde{F}_{\mu\nu} \underset{\substack{(12)\\(14)}}{=} \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} \underset{\substack{(12)\\(14)}}{=} \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \quad (8)$$

$$\tilde{F}_{\mu\nu} \underset{(13)}{=} \left(\vec{B}, -\vec{E}\right)$$

$$\tilde{F}^{\mu\nu} \underset{(13)}{=} \left(-\vec{B}, -\vec{E}\right) \quad (9)$$

$$\epsilon^{0123} = 1 \qquad \epsilon_{0123} = -1 \quad (10)$$

$$\epsilon^{\mu\nu\lambda\rho} = \begin{cases} 1 & \text{even } (0123) \\ -1 & \text{odd } (0123) \\ 0 & \text{otherwize} \end{cases} \quad (11)$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \quad (12)$$

$$\tilde{F}_{\mu\nu} \underset{(12)}{=} (\vec{B}, -\vec{E}) \qquad \tilde{F}^{\mu\nu} \underset{(12)}{=} (-\vec{B}, -\vec{E}) \quad (13)$$

$$\left\{\begin{array}{l}
\tilde{F}_{01} \underset{(11,12)}{=} -F^{23} = B_x \\[2ex]
\tilde{F}_{12} \underset{(11,12)}{=} \underbrace{\epsilon_{1203}}_{\epsilon_{0123}=-1} F^{03} = -(-)E_z = E_z \\[4ex]
\tilde{F}^{01} \underset{(11,12)}{=} F_{23} = -B_x \\[2ex]
\tilde{F}^{23} \underset{(11,12)}{=} \underbrace{\epsilon^{2301}}_{+1} F_{01} = E_x
\end{array}\right. \qquad (14)$$

$$\frac{1}{2} F^{\mu\nu} F_{\mu\nu} \underset{(6)}{=} \vec{E}^2 - \vec{B}^2 = -\frac{1}{2} \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} \quad (15)$$

$$\frac{1}{2} \tilde{F}^{\mu\nu} F_{\mu\nu} \underset{(8)}{=} \vec{E} \cdot \vec{B}$$

$$\left\{ \begin{array}{lll} \vec{E}^2 - \vec{B}^2 = \text{inv} & & \text{scalar} \\ & (15) & \\ \vec{E} \cdot \vec{B} = \text{inv} & & \text{pseudscalar} \end{array} \right. \quad (16)$$

$$P : \left\{ \begin{array}{ll} \vec{E} \to -\vec{E} & \text{vector} \\ \vec{B} \to +\vec{B} & \text{pseudovector} \end{array} \right.$$

$$T : \left\{ \begin{array}{l} \vec{E} \to \vec{E} \\ \vec{B} \to -\vec{B} \end{array} \right.$$

First pair of Maxwell's eqs

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \end{cases} \tag{17}$$

Solution
$$\begin{cases} \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{E} = -\vec{\nabla}V - \frac{1}{c}\dot{\vec{A}} \end{cases} \tag{18}$$

$(3, 10, 12) \Rightarrow$

$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho} \\ \epsilon^{01234} = -\epsilon_{01234} = 1 \\ \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} \end{cases}$$

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0} \tag{19}$$

$$\Updownarrow$$

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \end{cases}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \left(\partial_\lambda A_\rho - \partial_\rho A_\lambda \right)$$

$$= 0 \tag{20}$$

Second pair of Maxwell's equations

$$\left\{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \\ \vec{\nabla}\times\vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{1}{c}\vec{j} \end{aligned}\right. \qquad (21)$$

$$\Updownarrow \quad \text{— see (23)}$$

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad (22)$$

$$\left\{\begin{aligned} &\nu = 0 \qquad \partial_m \underbrace{F^{m0}}_{E_m} = \rho \\ &\nu = n \qquad -\frac{1}{c}\frac{\partial}{\partial t}\vec{E}_n + \vec{\nabla}\times\vec{B} = \frac{1}{c}\vec{j} \end{aligned}\right. \qquad (23)$$

$$\left\{\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}\right. \quad \text{Maxwell's eqs} \tag{24}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{25}$$

$$\underbrace{\partial_\mu \partial^\mu}_{\partial^2} A^\nu - \partial^\nu (\partial_\mu A^\mu) = j^\nu \tag{26}$$

Lorentz gauge condition

$$\partial_\mu A^\mu = 0 \tag{27}$$

$$\boxed{\partial^2 A^\mu = j^\mu} \tag{28}$$

$$\partial^2 \equiv \partial_\mu \partial^\mu$$

Coulomb law $\qquad \vec{E} = \dfrac{e}{4\pi r}\,\vec{n}$ (29)

$$\Downarrow$$

$$-\Delta V \underset{(29)}{=} \rho \qquad (30)$$

Lorentz ind

$$-\Delta \Rightarrow \partial^2 \qquad (31)$$

Lorentz ind $\qquad \rho \Rightarrow j^\mu$ (32)

$$\Downarrow$$

$$V \underset{(30,32)}{\Rightarrow} A^\mu \qquad (33)$$

$$\boxed{\partial^2 A^\mu \underset{(30,31,32,33)}{=} j^\mu}$$

Gauge inv

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu f \qquad (34)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow$$

$$\rightarrow F'_{\mu\nu} = \partial_\mu \left(A_\nu + \partial_\nu f\right) - \partial_\nu \left(A_\mu + \partial_\mu f\right)$$

$$= F_{\mu\nu} + \underbrace{\left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right)}_{0} f \qquad (35)$$

$$\left\{\begin{array}{l} F_{\mu\nu} \ (35) \ \text{gauge inv} \\ A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \varphi \end{array}\right. \qquad (36)$$

$$\left\{\begin{array}{l} V \rightarrow V' = V + \frac{1}{c}\dot{f} \\ \vec{A} \rightarrow \vec{A}' = \vec{A} - \vec{\nabla} f \end{array}\right. \qquad (37)$$

Equation of motion for a charged particle

Nonrelativistic case

$$m\vec{a} = e\vec{E} \qquad (38)$$

Consider some moment of time t_0

Take a reference frame

$$\vec{V}(t_0) = 0 \qquad (39)$$

$$m\frac{d}{dt}\vec{V} = e\vec{E} \qquad t = t_0 \qquad (40)$$

$$m \frac{d\vec{V}}{dt}^{n} = \frac{d}{\sqrt{1-\frac{v^2}{c^2}}\,dt} \left(\frac{m\vec{V}}{\sqrt{1-v^2/c^2}} \right)^{n} =$$

$$\underset{(39)}{=} c \frac{d}{ds} p^{n} \qquad (41)$$

$$u^{\mu} \underset{(39)}{=} (1, \vec{0}) \qquad (42)$$

$$\left(\vec{E}\right)^{n} \underset{(42)}{=} F^{n\mu} u_{\mu} = F^{n0}$$

$$c \frac{dp^{n}}{ds} \underset{(40, 41, 42)}{=} e F^{n\mu} u_{\mu} \qquad (43)$$

$$\underbrace{c \frac{dp^{0}}{ds}}_{0} = \underbrace{F^{0\mu} u_{\mu}}_{0} \qquad (44)$$

$$c\,\frac{dp^\mu}{dS} = eF^{\mu\nu}u_\nu \qquad (45)$$

(43, 44)

$$\begin{cases} t = t_0 \\ V(t_0) = 0 \end{cases}$$ spesial coordinate system

Lorentz in σ $\Rightarrow$ (46)

$$\boxed{c\,\frac{dp^\mu}{dS} \underset{45,46}{=} eF^{\mu\nu}u_\nu \qquad \text{Always}} \qquad (47)$$

$$\frac{d}{dt}\left(\frac{m\vec{V}}{\sqrt{1-v^2/c^2}}\right) \tag{47}= e\underbrace{\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)}_{\vec{F}} \quad (48)$$

$$\vec{F} = \vec{F_\ell} + \vec{F_t} \quad (49)$$

$$\begin{cases} \vec{F_\ell} = \dfrac{\vec{v}\,(\vec{v}\cdot\vec{F})}{v^2} \\ \vec{F_t} = \vec{F} - \dfrac{\vec{v}\,(\vec{v}\cdot\vec{F})}{v^2} \end{cases} \quad (50)$$

$$\begin{cases} m\gamma\,\dfrac{d\vec{V_t}}{dt} = e\,\vec{F_t} \\ m\gamma^3\,\dfrac{d\vec{V_\ell}}{dt} = e\,\vec{F_\ell} \end{cases} \quad (48) \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} \quad (51)$$

Examples:

$$\begin{cases} \bar{E} = \text{Const} \\ B = 0 \end{cases} \qquad E^2 > B^2 \qquad (52)$$

$$\begin{cases} \vec{B} = \text{Const} \\ \vec{E} = 0 \end{cases} \qquad E^2 < B^2 \qquad (53)$$

$$\begin{cases} E = B \\ \vec{E} \perp \vec{B} \end{cases} \qquad (54)$$

Plane wave : $\begin{cases} \text{Linear polarization} \\ \text{Circular polarization} \end{cases}$ (55)