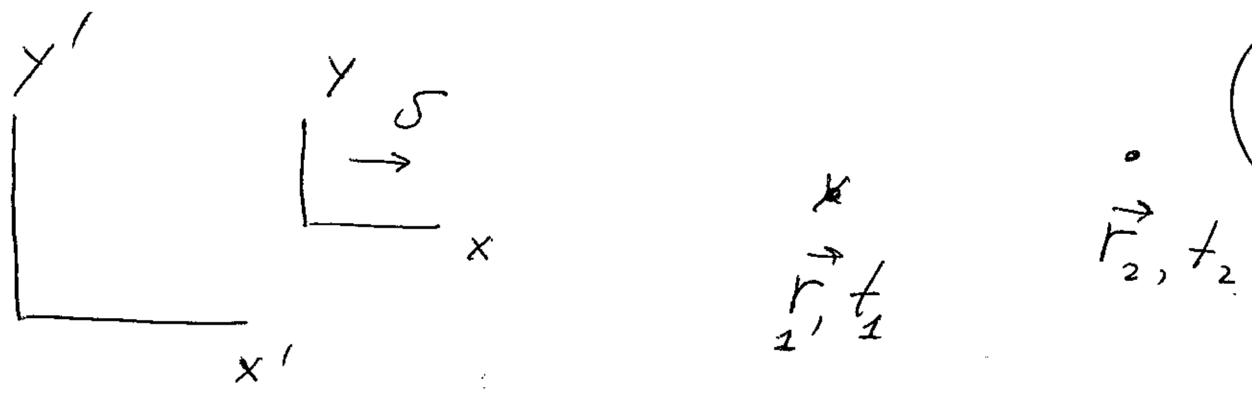
Loretz invariance,

4 - vectors etc

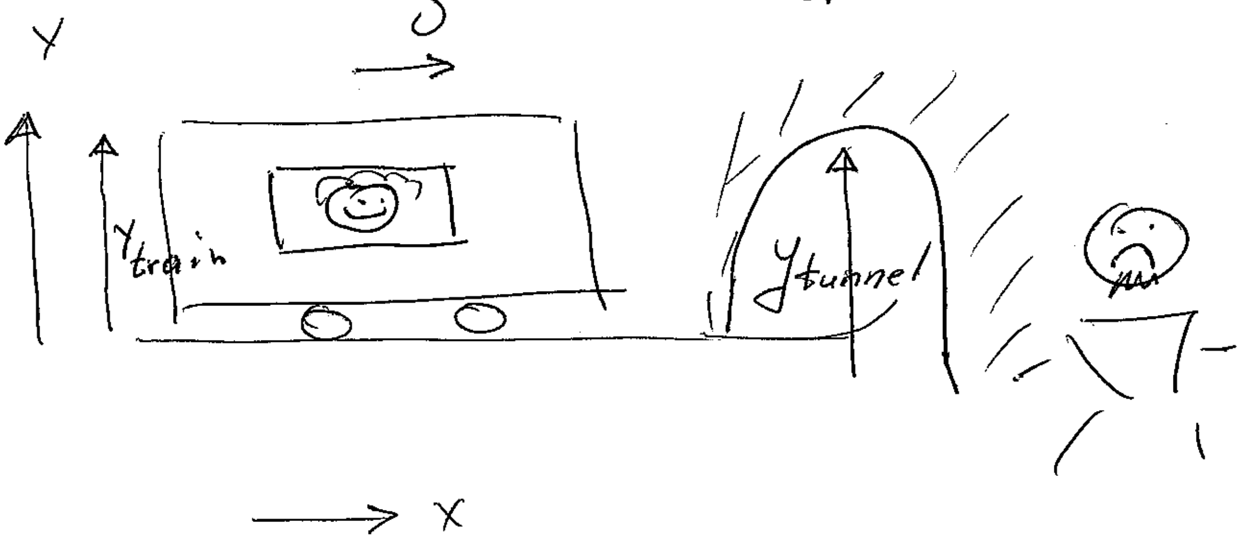
1. Introduction

$$\left\{ \begin{array}{l} c = \text{Const} \;! \\ \\ c \equiv 1 \end{array} \right. \qquad (1)$$



(2)

$$\left\{ \begin{array}{l} \Delta t = t_2 - t_1 \neq \Delta t' = t_2' - t_1' \\ \Delta x = x_2 - x_1 \neq \Delta x' = x_2' - x_1' \\ \Delta y = \Delta y' \\ \Delta z = \Delta z' \end{array} \right. \qquad (3)$$



"Theorem"

Assume $Y_{train} = Y_{tunnel}$ (4)

for $v = 0$.

Then $Y_{train} = Y_{tunnel}$ (5)

for $v > 0$.

Prove: assume otherwise, for example

$Y_{train} > Y_{tunnel}$ (6)

from the point of view of

Them $Y_{train} < Y_{tunnel}$ from the (7)
point of view of

It is impossible that simulteneously

 sees $\gamma_{train} > \gamma_{tunnel}$

(disaster)

 sees $\gamma_{train} < \gamma_{tunnel}$

(no problem)

$$\Downarrow$$

$$\gamma_{train} = \gamma_{tunnel}$$

2. Time:

(8)

$$\begin{cases} \Delta t\, c = \Delta y & (9) \\ \Delta t' c = \sqrt{\Delta y'^2 + v^2 \Delta t'^2} \end{cases}$$

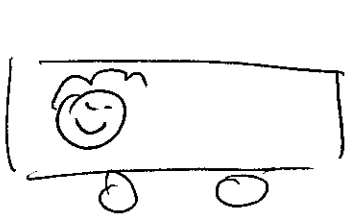
$$\Delta t'^2 (c^2 - v^2) \underset{(9)}{=} \Delta y'^2 \qquad (10)$$

$$\begin{cases} \Delta y \underset{(9,10)}{=} c\, \Delta t & (11) \\ \Delta y' = \sqrt{c^2 - v^2}\, \Delta t' \end{cases}$$

$$\Delta t' \underset{(5,11)}{=} \frac{\Delta t}{\sqrt{1 - v^2/c^2}} \qquad (12)$$

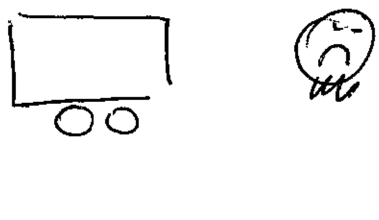
$$\Delta t' > \Delta t \qquad (13)$$

$$\left\{ \begin{array}{l} \sqrt{1 - v^2/c^2} \equiv \sqrt{1 - v^2} \\ \gamma = \dfrac{1}{\sqrt{1 - v^2}} \end{array} \right.$$

3. Length (see below) 

$$\ell' = \sqrt{1 - v^2}\, \ell \qquad (14)$$

$$\ell' < \ell$$



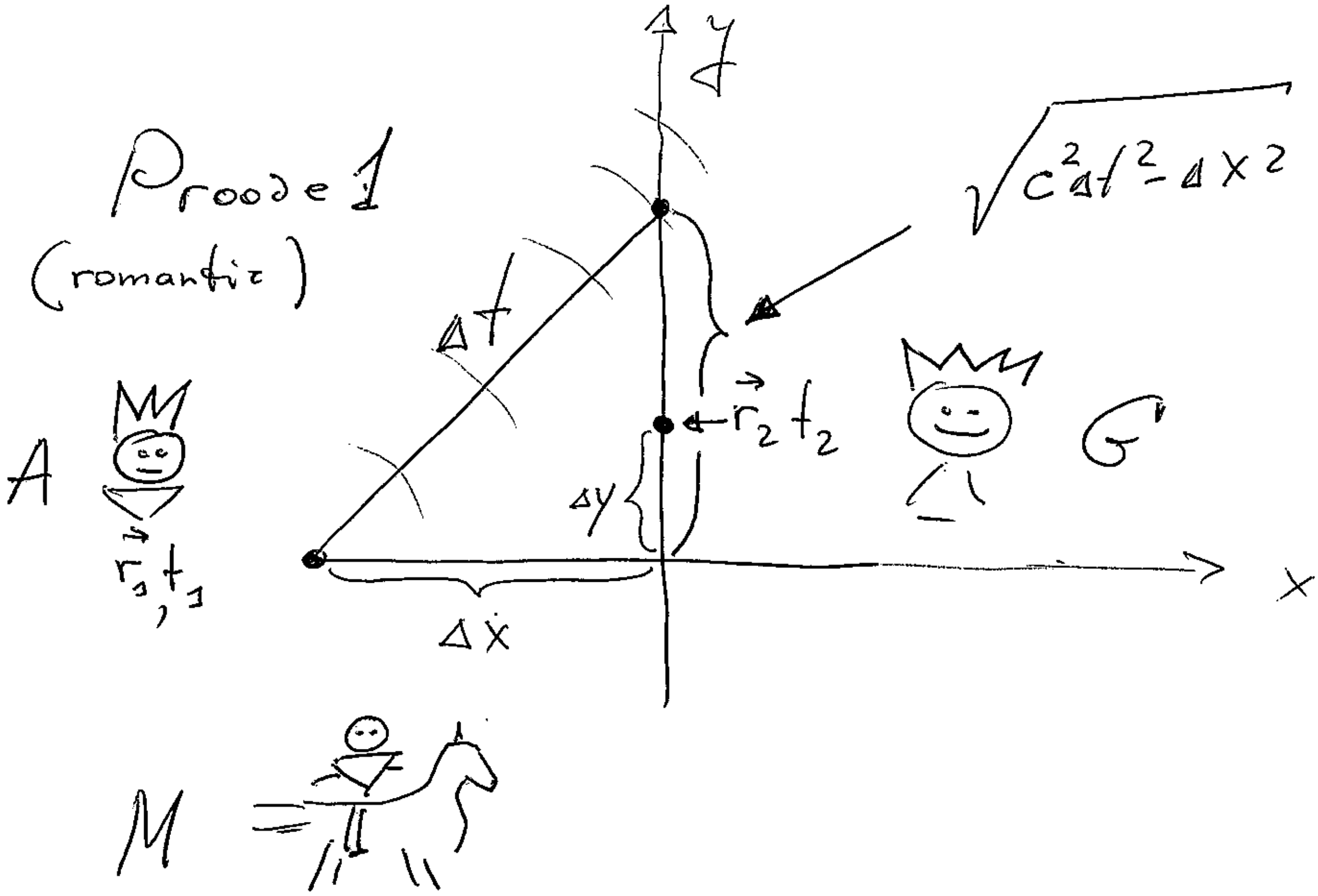
4. Interval

$$\Delta S^2 = c^2 \Delta t^2 - \Delta r^2 \qquad (15)$$

Theorem $$\Delta S = \Delta S' \qquad (16)$$

Proode 1
(romantic)

$$\sqrt{c^2\Delta t^2 - \Delta x^2} \qquad (17)$$



$A \equiv$ Arthur

$G \equiv$ Guinever

$M \equiv$ Merlin

$$(17) \Rightarrow \begin{cases} \sqrt{c^2 \Delta t^2 - \Delta x^2} = ind & (18) \\ \Delta y = ind \\ \Delta z = ind \end{cases}$$

$$c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = ind \quad (19)$$

$$(19) \Rightarrow (16)$$

Proove 2

Scientific

Assume that $\Delta S = 0$ (20)

Then $c = \frac{\Delta r}{\Delta t} = c' = \frac{\Delta r'}{\Delta t'}$

$$\Delta S = \Delta S' = 0 \quad (21)$$

Consider small interval $\Delta S \Rightarrow dS$

$$\begin{cases} dS = a\, dS' \\ a = a(v) \end{cases} \qquad (22)$$

Consider three systems K K_1 K_2

$$(22) \Rightarrow \begin{cases} dS^2 = a(V_1)\, dS_1^2 \\ \quad\;\; = a(V_2)\, dS_2^2 \\ dS_1^2 = a(V_{12})\, dS_2^2 \end{cases} \qquad (23)$$

$$\frac{a(V_2)}{a(V_1)} \underset{(23)}{=} a(V_{12}) \qquad (24)$$

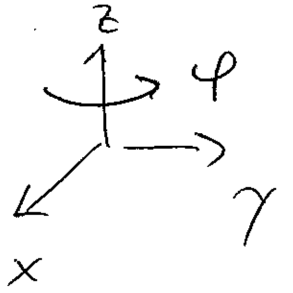
$$V_{12} = |\vec{V_1} - \vec{V_2}| \qquad (25)$$

$$a \overset{(24,25)}{=} 1 \qquad (26)$$

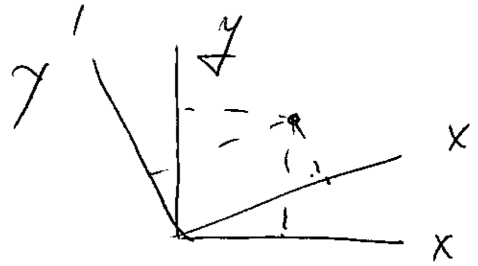
5 Lorentz transformation

$$S^2 = c^2t^2 - r^2$$

Rotation :

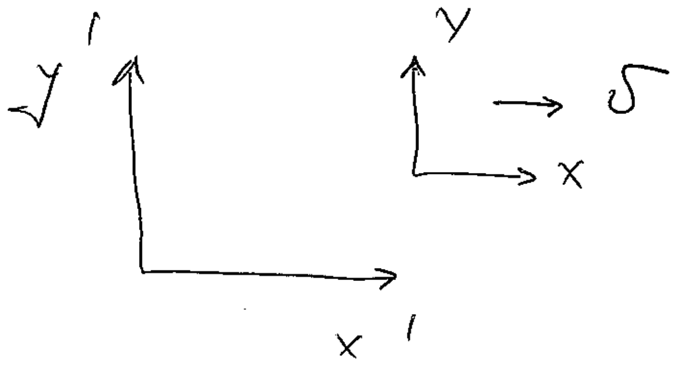


$$\begin{cases} t' = t \\ x' = \cos\varphi\, x + \sin\varphi\, y \\ y' = -\sin\varphi\, x + \cos\varphi\, y \end{cases} \qquad (27)$$



$$z' = z$$

Proper Lorentz transformation



$$\begin{cases} ct' = \cosh\psi \, ct - \sinh\psi \, x \\ x' = -\sinh \, ct + \cosh\psi \, x \\ y' = y \\ z' = z \end{cases} \qquad (28)$$

Clock

$$\Delta X = 0 \qquad (29)$$

$$c\,\Delta t' \underset{(28,29)}{=} \cosh\psi \; c\,\Delta t$$

$$\Delta t' = \cosh\psi\,\Delta t = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} \qquad (30)$$

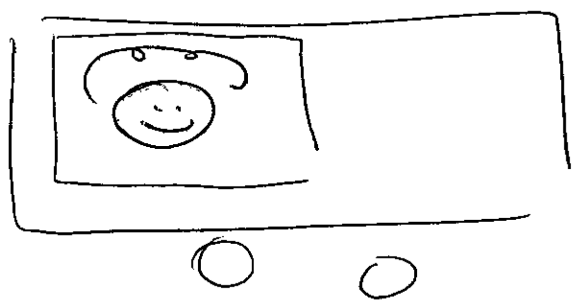
$$\begin{cases} \cosh\psi \underset{(30)}{=} \dfrac{1}{\sqrt{1-v^2/c^2}} \\ \sinh\psi = \dfrac{v}{\sqrt{1-v^2/c^2}} \end{cases} \qquad (31)$$

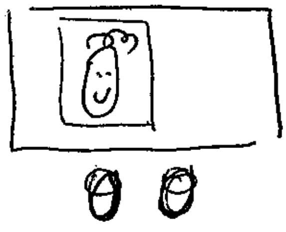
$$\begin{cases} ct' = \dfrac{ct - vx}{\sqrt{1 - v^2/c^2}} \\ x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}} \end{cases} \quad (32)$$

Verify that

$$\ell = \sqrt{1 - v^2/c^2}\, \ell_0 \quad (33)$$

i.e. that

the car 

looks like  for 

6. 4 - vectors

Contravariant vector

$$\left\{\begin{array}{l} x^\mu = (ct, x, y, z) \equiv \\ \quad = (ct, \vec{r}) \\ \mu = 0, 1, 2, 3 \\ x_\mu = (ct, -\vec{r}) \quad \text{covariant vector} \end{array}\right. \qquad (34)$$

$$x_\mu x^\mu \equiv \sum_{\mu=0}^{3} x_\mu x^\mu \equiv \sum_{\mu,\nu} x^\mu x^\nu g_{\mu\nu}$$

$$\equiv x^\mu x_\mu \equiv \sum_{\mu,\nu} x_\mu x_\nu g^{\mu\nu} \qquad (35)$$

$$x^\mu = g^{\mu\nu} x_\nu \equiv \sum_\mu g^{\mu\nu} x_\nu$$

$$x_\mu = g_{\mu\nu} x^\nu$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (36)$$

Same for $g_{\mu\nu}$

Notation $g^{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$

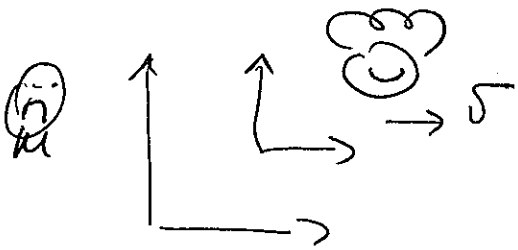
$\equiv (+, -, -, -)$

$$\Delta s^2 = x_\mu x^\mu \tag{37}$$

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \tag{38}$$

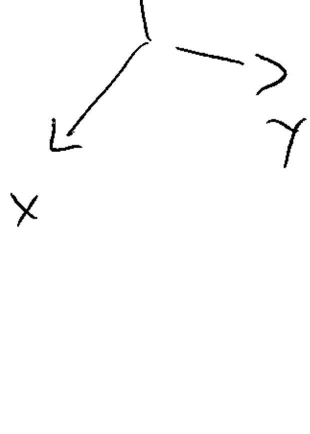
(39)

Examples:



$$\Lambda^\mu{}_\nu \underset{(28)}{=} \begin{pmatrix} \cosh\psi & -\sinh\psi & 0 & 0 \\ -\sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation



$$\Lambda^\mu{}_\nu \underset{(27)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu \qquad (41)$$

$$\Lambda_\mu{}^\nu \underset{(38,41)}{=} g_{\mu\mu'}\, g^{\nu\nu'}\, \Lambda^{\mu'}{}_{\nu'} \qquad (42)$$

$$\left\{ \begin{array}{l} z^\mu = x^\mu + y^\mu \\ z^2 \equiv z^\mu z_\mu = \underbrace{x^2}_{x^\mu x_\mu} + 2\, x^\mu y_\mu + \underbrace{y^2}_{y^\mu y_\mu} \\ z^2,\ x^2,\ y^2 \ \text{are inv} \end{array} \right.$$

$$x^\mu y_\mu = \text{inv} \qquad (43)$$

If S is inv

then $\partial_\mu S = \dfrac{\partial S}{\partial x^\mu}$ is a covariant vector (44)

$\partial^\mu S = \dfrac{\partial S}{\partial x_\mu}$ is a contravariant vector (45)

$$\left\{\begin{array}{l} x^\mu = (t, \vec{r}) \\ x_\mu = (t, -\vec{r}) \end{array}\right. \quad ! \qquad (46)$$

$$\left\{\begin{array}{l} \partial_\mu = (\partial_t, \vec{\nabla}) \\ \partial^\mu \overset{(44,45)}{=} (\partial_t, -\vec{\nabla}) \end{array}\right.$$

7. Energy – momentum

Some wave propagates

$kx - \omega t$ – phase =

$$\Rightarrow k_x x + k_y \cdot y + k_z z - \omega t = \vec{k}\cdot\vec{r} - \omega t \qquad (47)$$

$$\begin{cases} \vec{p} = \hbar \vec{k} \\ \varepsilon = \hbar \omega \end{cases} \qquad (48)$$

$$\text{Phase} = \frac{1}{\hbar}\left(\vec{p}\cdot\vec{r} - \varepsilon t\right) \equiv \frac{S}{\hbar} \qquad (49)$$

$$S' \; - \; \text{inv} \tag{50}$$

$$\left.\begin{aligned} \vec{p} &= \vec{\nabla} S \\ \frac{1}{c}\,\mathcal{E} &= \frac{\partial S}{c\,\partial t} \end{aligned}\right\} \tag{51}$$

$$p^\mu \underset{(51)}{=} \partial^\mu S \tag{52}$$

$$\boxed{p^\mu \underset{52,45}{=} \left(\frac{\mathcal{E}}{c}, \vec{p}\right)} \tag{53}$$

4 - vector

$$S' \underset{(49,53)}{=} - p^\mu x_\mu$$

$(53) \Rightarrow$

$$p^\mu p_\mu = \frac{\mathcal{E}^2}{c^2} - \vec{p}^{\,2} = \text{Const} \equiv \mu^2 c^2 \qquad (54)$$

$$\mathcal{E} \underset{(54)}{=} \sqrt{\mu^2 c^4 + c^2 p^2} \qquad (55)$$

$$\begin{cases} p \to 0 \\ \mathcal{E} \underset{(55)}{=} \mu c^2 + \underbrace{\frac{p^2}{2\mu}}_{} \end{cases} \qquad (56)$$

$$\boxed{\mu = m} \qquad (57)$$

$$\begin{cases} p^\mu p_\mu = m^2 c^2 \\ \mathcal{E}^2 - c^2 p^2 \underset{(55,57)}{=} m^2 c^4 \end{cases} \qquad (58)$$

$$p = 0 \qquad \boxed{\mathcal{E} = mc^2} \qquad (59)$$

8. Velosity

$$ds = dx^\mu dx_\mu = c^2 dt^2 - d\vec{r}^{\,2} =$$

$$= c\, dt \sqrt{1 - \frac{v^2}{c^2}} \qquad (60)$$

$$u^\mu \underset{(60)}{=} \frac{dx^\mu}{ds} = \qquad (61)$$

$$= \frac{1}{\sqrt{1 - v^2/c^2}} \left(1, \vec{v}/c \right)$$

$$u^\mu u_\mu \underset{(61)}{=} 1 \qquad (62)$$

9. Current

$$\rho \,,\quad \vec{j}$$

$$\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0 \tag{63}$$

$$j^{\mu} \underset{(63)}{=} \left(c\rho \,,\; \vec{j} \right) \tag{64}$$

19. 4-Potential

$$V, \vec{A} \quad \Rightarrow \quad A^\mu = (V, \vec{A}) \qquad (63)$$

$$\begin{cases} \varepsilon \xrightarrow{EM} \mathcal{E} = \varepsilon + eV \\ \vec{p} \longrightarrow \vec{\mathcal{P}} = \vec{p} + e\vec{A} \end{cases}$$