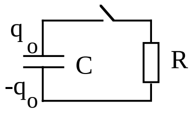
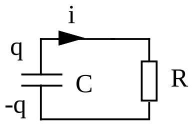
Notes on Inductance and Circuit Transients

Joe Wolfe, Physics UNSW

Circuits with R and C



What happens when you close the switch?
(close switch at t = 0)



Current flows *off* capacitor, so

$$i = -\frac{dq}{dt}$$

Across capacitor: $V_C = \frac{q}{C}$ $\qquad C \equiv \frac{q}{V}$

Across resistor: $V_R = iR = V_c$

$$\therefore \quad \frac{dq}{dt} = -i$$

$$= -\frac{V}{R}$$

$$= -\frac{q}{RC}$$

so $$\frac{dq}{q} = -\frac{1}{RC}\,dt$$

$$\int \frac{dq}{q} = -\frac{1}{RC}\int dt$$

$$\mathit{ln}\, q = -\frac{t}{RC} + \text{const.}$$

$t = 0,\ q = q_o \quad \therefore \quad \mathit{ln}\, q_o = \text{const.}$

$$\mathit{ln}\, q - \mathit{ln}\, q_o = -\frac{t}{RC} \qquad \mathit{ln}\, a - \mathit{ln}\, b = \mathit{ln}\,\frac{a}{b}$$

$$\ln \frac{q}{q_o} = -\frac{t}{RC}$$

$$\therefore \quad q = q_o\, e^{-t/RC} \qquad V = q/c \ \therefore$$

$$V = V_o e^{-t/RC} \qquad (1)$$

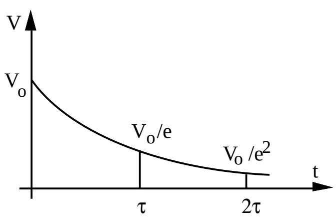
$$\tau = RC = \text{time constant}$$

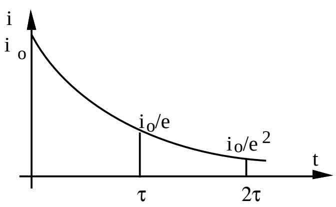
Units: $\Omega F = \frac{V}{A} . \frac{C}{V} = \frac{s}{C} . C = s$

$$i = -\frac{dq}{dt}$$

$$= \frac{q_o}{RC} e^{-t/RC} = \frac{V_o}{R} e^{-t/RC}$$

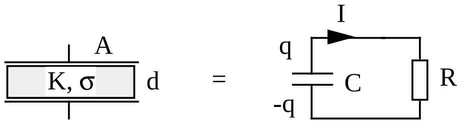
$$i = i_o\, e^{-t/\tau}$$





Example

A real C uses dielectric with dielectric const. 4.0 and conductivity $\sigma = 2\ 10^{-15}\ \Omega^{-1}m^{-1}$. It is charged then left disconnected for 1 hour. What fraction of charge remains? ($\sigma \equiv 1/\rho$)



$$C = \frac{\varepsilon A}{d} = \frac{K\varepsilon_o A}{d}$$

$$R = \frac{\rho L}{A} = \frac{d}{A\sigma}$$

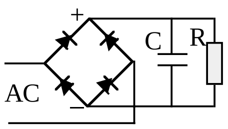
$$\therefore \quad RC = \frac{d}{A\sigma} \cdot \frac{K\varepsilon_o A}{d} = \frac{K\varepsilon_o}{\sigma}$$

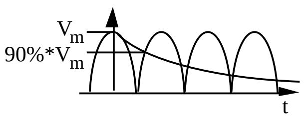
$$= \frac{4.0 \times 8.9\ 10^{-12}\ Fm^{-1}}{2.0\ 10^{-15}\ \Omega^{-1}m^{-1}} = 1.78 \times 10^4\ s$$

$$V = V_o e^{-t/RC}$$

$$\frac{Q}{Q_o} = \frac{V}{V_o} = e^{-t/RC} = e^{-3600/\tau} = 0.82$$

Example Electronic appliance requires 20 W at 40 V DC, with max variation of 10%. What capacitance needed to store energy between cycles?





AC has 50 cycles per second, $\therefore$ turns off 100 times/second, ie every 10 ms.

$$\frac{V}{V_m} = e^{-t/RC} \cong 0.90$$

t = 10 ms, C = ?, what is R?

R is the effective resistance of the circuit.

We know V and P, so use $P = \frac{V^2}{R}$

$$R = V^2/P = 80\ \Omega.$$

$$e^{-t/RC} \cong 0.90$$

$$\frac{t}{RC} \cong -\ln(0.90)$$

$$\therefore \quad C \cong \frac{-10\ ms}{80\ \Omega * \ln .9}$$

$$= 13000\ \mu F$$

Example the membrane of a "pacemaker" neurone of the heart has a capacitance of 10 pF and a conductance G = 3.0 pΩ^{-1}. Initially the potential difference across this membrane is –80 mV. The neurone begins a pulse when the potential difference reaches –60 mV. How long before it begins a pulse? Neglecting the duration of the electrical impulse itself, what is the pulse rate?

(1) $\rightarrow \quad \ln\frac{V}{V_o} = -\frac{t}{RC}$

$$t = -RC \ln\frac{V}{V_o} = -\frac{10\ 10^{-12}}{3\ 10^{-12}} \ln\left(\frac{-60}{-80}\right)$$

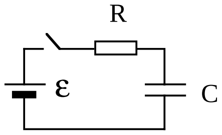
$$= 0.96\ s$$

The "leakiness" of the membrane is now suddenly increased by a factor of 2. How long between electrical impulses? and what is the approximate pulse rate, again neglecting the duration of the inidividual impulses.

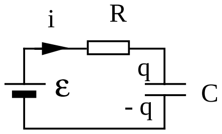
$$t = -R_2 C \ln\frac{V}{V_o} == -\frac{RC}{2} \ln\frac{V}{V_o} = ...$$

$$= 0.48\ s$$

RC charging



What happens when you close the switch?
(close switch at t = 0,
$q = 0$, $V = 0$)



Current flows *onto* capacitor, so

$$I = +\frac{dq}{dt}$$

Kirchoff $\quad \varepsilon = iR + \frac{q}{C}$

take derivatives, use $i = +\frac{dq}{dt}$

$\therefore \qquad 0 = R\frac{di}{dt} + \frac{i}{C}$

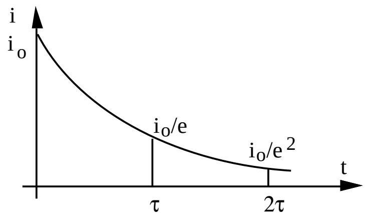
rearrange: $\quad \frac{di}{i} = -\frac{dt}{RC}$

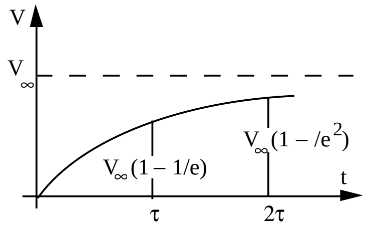
$\therefore \qquad \ln i = -\frac{t}{RC} + \text{const}$

initial condition $\Rightarrow$ $\ln i_o = \text{const}$

$$i = i_o e^{-t/\tau} = \frac{\varepsilon}{R}e^{-t/\tau} \qquad (2)$$

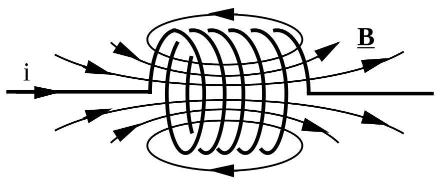
where $i_o = \varepsilon/R$ and $\tau = RC$





Inductance

Self-induction



flux through each turn of coil $\approx \phi_B$

Faraday $\quad \varepsilon = -\frac{d(N\phi_B)}{dt}$

Inductance, L $\qquad L \equiv \frac{N\Phi_B}{i} \qquad (3)$

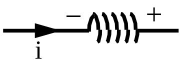
(L is the total flux linkage per unit current)

$$\text{So} \quad \varepsilon = -L\frac{di}{dt} \qquad (4)$$

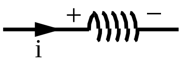
Note: $\quad \varepsilon$ in coil $\quad \propto$ rate of change in i
cf $\quad$ V across C $\quad \propto$ integral of i
Units: $\quad$ 1 henry = 1 Volt second/amp

Lenz's law:

ε acts in direction to maintain current (and $\therefore$ **B**)



if i increases ε $\leftarrow$

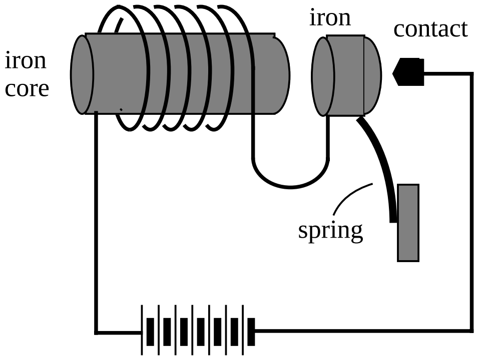


if i decreases ε $\rightarrow$

$\therefore$ L resists change in i

Example Automobile ignition coil (schematic)

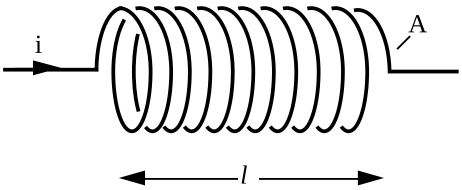
(not in syllabus)



spring moves iron right to close contact. i flows which causes **B**
B attracts iron to left, which opens circuit

$$\varepsilon = -L\frac{di}{dt} \qquad i \rightarrow 0 \text{ quickly so large } \varepsilon$$

Ideal Solenoid



n turns/m

$N\Phi_B = n\,l\,BA$

but $B = \mu_o ni$

$$\therefore \quad L = \frac{N\Phi_B}{i} = \frac{n l \mu_o n i A}{i}$$

$$L = \mu_o n^2 l A \qquad (5)$$

Note: for non-magnetic materials, $\mu \approx \mu_o = 4\pi\ 10^{-7}\ \mathrm{Hm^{-1}}$
magnetic materials, $\mu >> \mu_o$ $\therefore$ iron cores for large L.

Example Coil in kettle has R = 60 Ω. It is 20 cm long, has 1 turn/mm, and is 4 mm in diameter.

i) what is L?

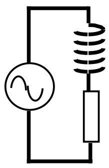
ii) it is connected to 240 V_{rms}. Switch has breakdown potential of 1000 V and is turned off at peak of cycle (V = 340 V). Estimate the duration of spark. (Neglect other L).

$$\text{Peak } i = \frac{V_p}{R} = \frac{340}{60} = 5.7\text{A.}$$

(5) $\rightarrow$ $L = \mu_o\, n^2\, l\, A$

$= 4\pi\ 10^{-7}\ .(10^3)^2\ .\ (0.2)\ .\ \pi(2\ 10^{-3})^2$ SI

$= 3.2\ \mu$H.



$$\varepsilon_L = -L\frac{di}{dt} \sim -1000 \text{ V during spark}$$

$$\therefore \frac{di}{dt} \sim \text{const.} = \frac{1000\text{ V}}{L}$$

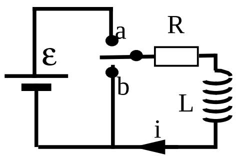
$$\therefore \Delta t \cong \frac{i}{\frac{di}{dt}} = \frac{5.7 \times 3.2\ 10^{-6}}{1000\text{ V}} = 18\text{ ns.}$$

(In practice Δt is larger because of other L's.)

R - L Transients

Recall $\varepsilon_L = - L \frac{di}{dt}$ (Lenz's law: back emf)

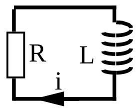
This prevents sudden change in i,
∴ i is continuous.



i) switch at a. After a long time,
$\frac{di}{dt} = 0, \quad i = \varepsilon_c/R$

ii) switch to b at t = 0

∴ at t = 0, $i = \varepsilon_c/R$



Kirchoff: $- L \frac{di}{dt} = iR$

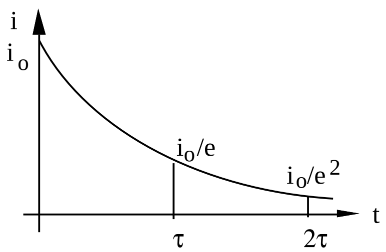
∴ $\frac{di}{i} = - \frac{R}{L} dt$

∫ → ∴ $\ln i = - \frac{Rt}{L} + \ln i_o$

∴ $i = i_o e^{-Rt/L} = \frac{\varepsilon_c}{R} e^{-t/\tau}$

where $\tau = L/R$ (6)

Units: $\frac{H}{\Omega} = \frac{V}{A/s} \cdot \frac{A}{V} = s$

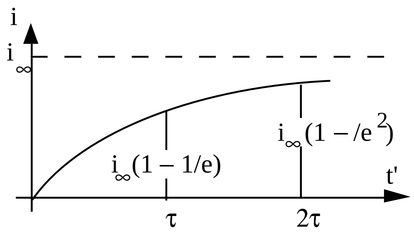


iii) switch at b for long time (i = 0). Switch to a at t′ = 0.

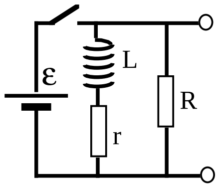
$\varepsilon_c - L \frac{di}{dt} = i R$

solve → $i = \frac{\varepsilon_c}{R} (1 - e^{- t'/\tau})$

where $\tau = L/R$ (7)



Example



ε = 6V
L = 264 H
r = 13k (≡ 13 kΩ)
R = 47k

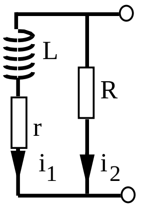
Switch closed for long time. Open at t = 0.
What does the CRO show?

Switch closed, after a long time $\frac{di}{dt} = 0$

Kirchoff's law: $V = i_1 r = i_2 R$
and $i_T = i_1 + i_2$

CRO shows $V = i_2 R = \varepsilon = 6$ V

Switch opened



L forbids abrupt change in i

$\therefore$ at t = 0 $\quad i_1(0) = \varepsilon/r$

$t \to \infty$, $i_1 \to 0$

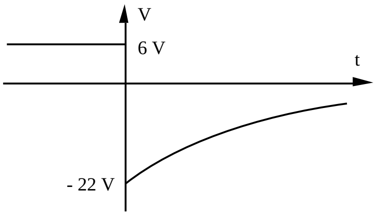
total series resistance is r+R

so $i_1 = \left(\frac{\varepsilon}{r}\right) e^{-t(r+R)/L}$

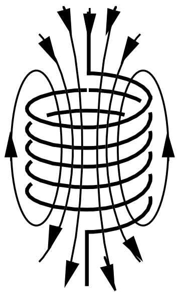
here $\tau = \frac{L}{r+R}$

$i_2 = -i_1$ $\quad$ so $V_{CRO} = i_2 R = -i_1 R = -\left(\frac{\varepsilon R}{r}\right)$

$$V_{CRO} = -\left(\frac{\varepsilon R}{r}\right) e^{-t(r+R)/L}$$



Energy in $\underset{\sim}{B}$ of L



Power supplied to L $= -\varepsilon_L i$

i.e. $\frac{dU_B}{dt} = \left(L \frac{di}{dt}\right) i$

$U_B = \int dU_B = \int_o^{i} Li'\, di'$

$U_B = \frac{1}{2} Li^2$ $\qquad$ (8)

c.f. $U_E = \frac{1}{2} CV^2$

Energy density

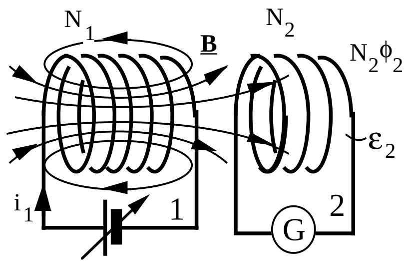
$$\frac{U_B}{vol} = \frac{U_B}{Al} = \frac{\frac{1}{2} Li^2}{Al}$$

(5) $\rightarrow$ $\quad L = \mu_o n^2 lA$ $\qquad$ (5)

(from mag.) $\rightarrow$ $\quad B = \mu_o in$

$\therefore$ $\quad \frac{U_B}{vol} = \frac{1}{2} \frac{B^2}{\mu_o}$ $\qquad$ (9) $\qquad$ (*c.f.* $U_E = \frac{1}{2} \varepsilon_o E^2$)

Mutual Induction



Mutual Inductance $\quad M_{21}$ (of 2 w.r.t. 1)

$M_{21} \equiv \frac{N_2 \Phi_{21}}{i_1}$ $\qquad$ $\left(c.f.\ L = \frac{N\Phi}{i}\right)$

so $\quad M_{21} i_1 = N_2 \Phi_{21}$

$M_{21} \frac{di_1}{dt} = N_2 \frac{d\Phi_{21}}{dt} = -\varepsilon_2$ $\qquad$ (Faraday's law)

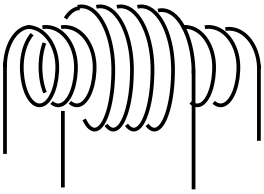
(long proof) $\Rightarrow$ $\quad M_{21} = M_{12} = M$

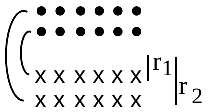
so: $\quad \varepsilon_2 = -M \frac{di_1}{dt}$, $\quad \varepsilon_1 = -M \frac{di_2}{dt}$ $\qquad$ (10)

(S.I. unit henry = Vs A^{-1})

Example

Two long concentric solenoids. $M = ?$





n_1 / unit length and n_2 / unit length

$$M = M_{21} = \frac{N_2 \Phi_{21}}{i_1} = \frac{n_2\, l\, \Phi_{21}}{i_1}$$

$$B_1 = \mu_o\, i\, n, \text{ and } \Phi_{21} = b_1 A_2$$

$$\therefore \quad M = \frac{n_2\, l}{i_1} (\mu_o\, i_1 n_1)\, \pi\, R_2^2$$

$$M = \pi R_2^2\, l\, \mu_o\, n_1\, n_2 \qquad (11)$$

LC oscillations

(assume ideal L: $r_L = 0$, ideal C: $r_C = \infty$)

Kirchoff's law:

$$- L \frac{di}{dt} = \frac{q}{C}$$

$$i = \frac{dq}{dt}, \qquad \text{so } - L \frac{d^2 i}{dt^2} = \frac{1}{C}\frac{dq}{dt} = \frac{i}{C}$$

$$\therefore \quad \frac{d^2 i}{dt^2} = - \frac{1}{LC}\, i = - \omega^2\, i \qquad \text{say}$$

solve: $i = \text{const} \sin(\omega t + \text{const})$

$$i = i_m \sin(\omega t + \phi)$$

$$\text{where } \omega = \sqrt{\frac{1}{LC}} \qquad (12)$$

$$V = - L \frac{di}{dt} = - L\, \omega\, i_m \cos(\omega t + \phi)$$

$$V = - V_m \cos(\omega t + \phi)$$

Energy

$$U_{E.} = \tfrac{1}{2} C V^2 = \frac{C}{2} [....\cos....]^2$$

$$U_B = \frac{1}{2} L i^2 = \frac{L}{2} [....\sin....]^2$$

$$U_T = U_E + U_B$$

$$= \frac{1}{2} i_m^2 L \left[LC\, \omega^2 \cos^2(\quad) + \sin^2(\quad)\right]$$

$$= \frac{1}{2} i_m^2 L \qquad [= \text{constant}]$$

also $V_m = i_m\, \omega L$

$$\text{so } U_T = \frac{1}{2} \frac{V_m^2}{\omega^2 L^2} L = \frac{1}{2} C V_m^2$$

Example

Front end of a radio: C_{max} $C_{min} = 1.1$ pF

$$C_{max}\,(3 \text{ in } //) = 3\left(\frac{\varepsilon_o A}{d}\right) = 3\, \frac{\varepsilon_o \pi r^2}{d} = ... = 16.7 \text{ pF}$$

ii) If we want to tune the AM band (54 - 160 kHz) what values of L and C_1? How do we make the L?

$$f_{max} = \frac{1}{2\pi}\, \omega_{max} = \frac{1}{2\pi} \sqrt{\frac{1}{L(C_{min} + C_1)}} \qquad f_{min} = \qquad = \frac{1}{2\pi} \sqrt{\frac{1}{L(C_{max} + C_1)}}$$

$$\therefore \quad \frac{f_{max}^2}{f_{min}^2} = ... = \frac{C_{max} + C_1}{C_{min} + C_1}$$

$$\therefore \; C_1 (f_{max}^2 - f_{min}^2) = C_{max} f_{min}^2 - C_{min} f_{max}^2$$

$$\therefore \quad C_1 = ... = 0.9 \text{ pF}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{1}{LC}} \quad \therefore \; L = \frac{1}{C} \cdot \frac{1}{4\pi^2 f^2} = \frac{1}{C_{max} + C_1} \cdot \frac{1}{4qp^2 f_{min}^2}$$

$$= 490 \text{ mH}$$

(5) $L = \mu_o n^2 lA$ say $n = 2000$/m , $A = 5$ cm^2, $l = ?$ $l = \frac{L}{\mu_o n^2 A} = 200$ m. Oops, try iron core: $\mu_r = \frac{\mu}{\mu_o} \sim 2000$