PHYS1221-1231 Relativity.

Notes for Physics and Higher Physics 1b. Joe Wolfe

See also our web pages:

http://www.phys.unsw.edu.au/~jw/time.html

http://www.phys.unsw.edu.au/~jw/relativity.html

http://www.phys.unsw.edu.au/~jw/twin.html

Theories of relativity

status

Galilean or Newtonian relativity
vector addition of velocities
(familiar. Common sense?)

usually an excellent approximation but wrong in extreme cases

Special theory of relativity (Einstein)
theory of **dynamics** including
uniform relative motion

in excellent agreement with a wide range of experiments

General theory of relativity (Einstein)
theory of **gravitation**, includes
dynamics in accelerated frames

works, but has competition tests have not been so precise or severe

'Cosmological relativity'
theory of evolution of the universe,
includes expansion of space

topical

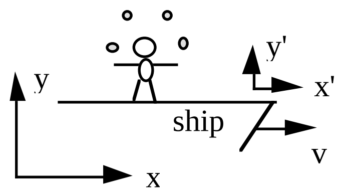
Special relativity

General Relativity → gravitation

Define: In an **inertial frame,** Newton's laws hold

why? Mach's principle?

Galilean Relativity (Gallileo – Newton – 1904)



$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilean transformation

$$u' = \frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{dx}{dt} - v = u - v \quad \textit{or} \quad u = v + u'$$

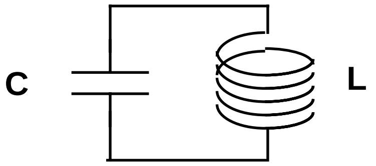
additivity of velocities

$$a' = \frac{d^2x'}{dt'^2} = \frac{du'}{dt} = \frac{du}{dt} = a$$

∴ if one frame is inertial, another with constant, uniform relative velocity is also inertial.

Principle of Galilean relativity:
Mechanics is the same in two such frames.

Why only mechanics? What about electromagnetism?
Are electrical components different if moving?
Does the frequency of a tuned circuit change?



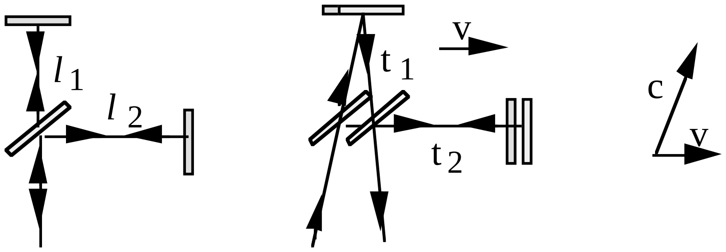
$$\omega = \sqrt{\frac{1}{LC}}$$

Capacitor: *Does* $C = \frac{\varepsilon_o A}{d}$?

Inductor: *Does* $L = \frac{\mu_o N^2 A}{l}$?

If so, is $\frac{1}{\sqrt{\varepsilon_o \mu_o}}$ *the same?*

Michelson & Morley



Suppose that light travels (at c) with respect to a 'stationary' medium (called the æther).
Set it up with $l_1 = l_2 = l$ and v = 0, light beams return in phase.
Now move at v with respect to æther, but let light travel at c in the æther (≠ c in lab).

$$t_2 = \frac{l}{c + v} + \frac{l}{c - v} = \frac{2l}{c(1 - v^2/c^2)}$$

$$t_1 = \frac{l}{\sqrt{c^2 - v^2}} + \frac{l}{\sqrt{c^2 - v^2}} = \frac{2l}{c\sqrt{1 - v^2/c^2}}$$

$$\Delta t = \ldots.. \cong 2l\frac{v^2}{c^3} \qquad l = 11 \text{ m}$$

Earth around sun: $v = 30 \text{ kms}^{-1}$

$$\Delta\phi = 2\pi\Delta t..\frac{c}{\lambda} = \ldots = 2.3 \text{ radians} \quad (0.4 \text{ fringes})$$

Result: 0.00 ± 0.01 fringes

What about electromagnetism?

Does $C = \frac{\varepsilon_o A}{d}$? *Does* $L = \frac{\mu_o N^2 A}{l}$? *If so, is* $\frac{1}{\sqrt{\varepsilon_o \mu_o}}$ *the same?*

Michelson and Morely's experiment:
c is the same in inertial frames

∴ velocities are not (exactly) additive *huh?*

Principle of special relativity:

Mechanics ***and Electromagnetism*** are the same in inertial frames.

Chose Maxwell's eqns ahead of additivity of velocity. μ_o, ε_o *same so c the same.*

Principle of special relativity:

Mechanics ***and Electromagnetism*** are the same in inertial frames.

this is in agreement with Michelson and Morely's experiment: c is the same in inertial frames

Common objection:

*But it **can't** be true, because then velocities wouldn't be additive. It's just not common sense. Consider the light from the headlights of a moving vehicle.*

EITHER i) it travels at c with respect to the ground, and so the driver must measures it to to go slower

But this is like M&M's experiment

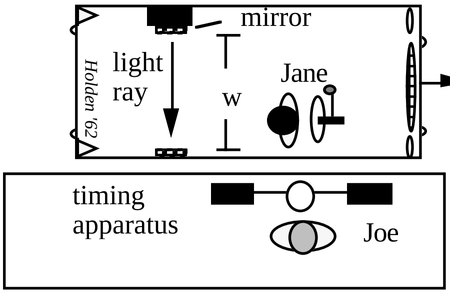
OR ii) the light from the headlights of the moving vehicle must travel at c+v

But we can measure light from double stars

Experimentally, velocites ***aren't*** additive, at least not when one of the velocities is of order c.

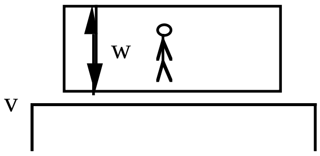
When v << c, velocites are almost exactly additive so we don't notice, so the slight difference never gets incorporated into common sense.

Most clocks are electromagnetic. Consider a very simple one: light beam going between two mirrors.



Jane is in the car
Joe is on the verandah

Most clocks are electromagnetic. Consider a very simple one: light beam going between two mirrors.



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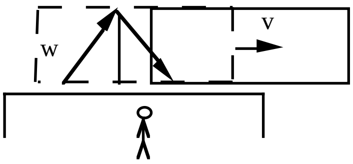
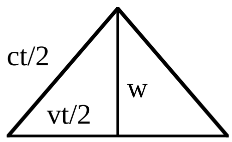




$t' = 2w/c \qquad t = 2\frac{\sqrt{w^2 + v^2(t/2)^2}}{c}$

$$c^2t^2 = 4\left(\frac{c^2t'^2}{4} + \frac{v^2t^2}{4}\right)$$

$$\therefore \quad t' = t.\sqrt{1 - \frac{v^2}{c^2}}$$

$t' = t/\gamma$ (*but from the vehicle...*)

Proper time t_o in the rest frame. (Here it is t')

$$\frac{\Delta t}{\Delta t_o} = \gamma \geq 1$$

in all other frames t is faster than proper time

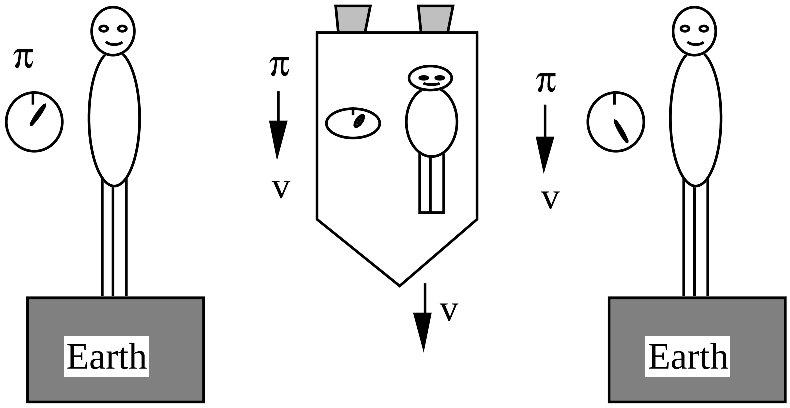
Tests: pions, clocks in aeroplanes, accelerators

Pion lifetimes

A slow pion in the lab has lifetime 2.2 μs

∴ proper lifetime of pion is 2.2 μs

A fast pion (w.r.t Earth) has lifetime 16 μs (Earth measurement)



Proper time $t' = t/\gamma$

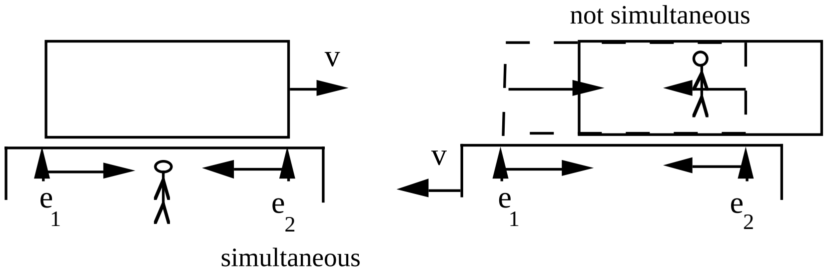
where t is the lifetime measured in another frame at v w.r.t. the pion.

(t always ≥ proper time).

$$\frac{t'}{t} = \sqrt{1 - \frac{v^2}{c^2}} \quad \therefore \quad v = 0.99\ c$$

Relativity of simultaneity and time-order

Events e_1 and e_2.



Two events that are simultaneous in one frame are not simultaneous in another

weird, but the difference is tiny

The order of occurrence can be different in two frames

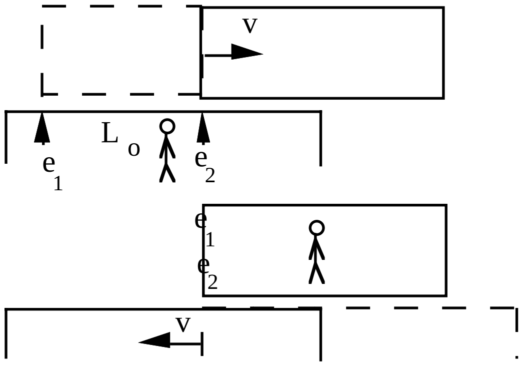
but only for events at (x,t) and (x+Δx,t+Δt)) where

$$\frac{\Delta x}{\Delta t} > c$$

Length measurements

Measure a length. If object moves, must measure simultaneously, or else compensate for motion.

Proper length L_o is length in the rest frame



$L_o = v\Delta t$ $L' = v\Delta t' = v\Delta t_o$

$$\frac{L'}{L_o} = \frac{v\Delta t_o}{v\Delta t} = \frac{1}{\gamma} \leq 1$$

Need new transformation equations. Try linear.

(Why?)

$x' = Ax + Bt$
$y' = y$ *(why?)*
$z' = z$
$t' = Ct + Dx$

At $x' = 0$, $x = vt$. $\therefore A = -B/v$

At $t = 0$, $x' = \gamma x = Ax$. $\therefore x' = \gamma(x - vt)$

At $x = 0$, $t' = \gamma t$ $\therefore C = \gamma$

At $x' = 0$, $t = \gamma t'$ & $x = vt$

$t/\gamma = \gamma t - Dvt$ $\therefore 1 = \gamma^2 - D\gamma v$

$$\therefore D = \frac{\gamma^2 - 1}{\gamma v}$$

Lorentz transformations

$x' = \gamma(x - vt)$ Check that $L/L_o = \gamma$
$y' = y$
$z' = z$
$t' = \gamma\left(t - \frac{v\,x}{c^2}\right)$ Check that $t/t_o = 1/\gamma$

For both, check that $v << c \Rightarrow$ Galileo

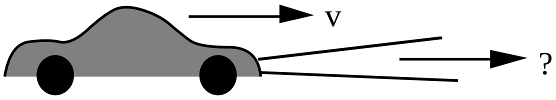
What about velocities?

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v\,dx}{c^2}\right)}$$

$$= \frac{\frac{dx}{dt} - v}{1 - \frac{v\,dx}{c^2 dt}} = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

or $$u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}}$$ *Non-additivity of velocities*

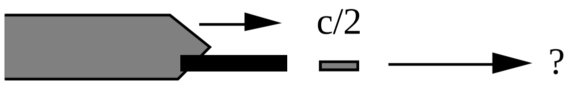
Example: how fast is the light from Jane's headlights? (Jane's car travels at v.)



$$u = \frac{u' + v}{1 + \frac{v u'}{c^2}}$$

$$= \frac{c + v}{1 + \frac{v c}{c^2}} = c$$

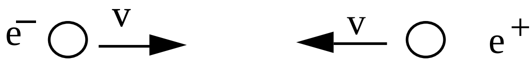
Example: the Vogon star ship travels towards earth at v = c/2. A Vogon fires a zap at Earth with speed of c/2 (w.r.t the ship). At what relative speed does the zap travel towards Earth?



$$u = \frac{u' + v}{1 + \frac{v u'}{c^2}}$$

$$= \frac{c/2 + c/2}{1 + \frac{(c/2)(c/2)}{c^2}} = \frac{c}{1 + 1/4} = 0.8\,c$$

Example: An electron travels* at 0.999 c in an accelerator. A postitron travels* at 0.999 c in the opposite direction. What is their relative speed?



$$u' = \frac{u - v}{1 - \frac{v u}{c^2}}$$

$$= \frac{0.999c + 0.999c}{1 + \frac{(0.999c)(0.999c)}{c^2}}$$

$$= \frac{1.998\ c}{1 + .998001} = 0.9999995\ c$$

**Warning: diagram and question misleading*

The twin paradox (≡ the clock paradox)

Ernest and Algernon Prism are twin babies. Ms Prism accidentally leaves Ernest in the baggage room of a space port where he is loaded onto the Ursa Major express (v = 0.99 c). The mistake discovered, he is transferred to a returning ship (0.99 c) when he is 35 light years away (as measured by Algernon on Earth).

Acclerating at say g,
Gen Rel effects are unimportant

When Ernest returns, Algernon has aged 70 years, but at $\gamma = 1/\sqrt{1 - 0.99^2} = 7$, Ernest has aged only 10 years.

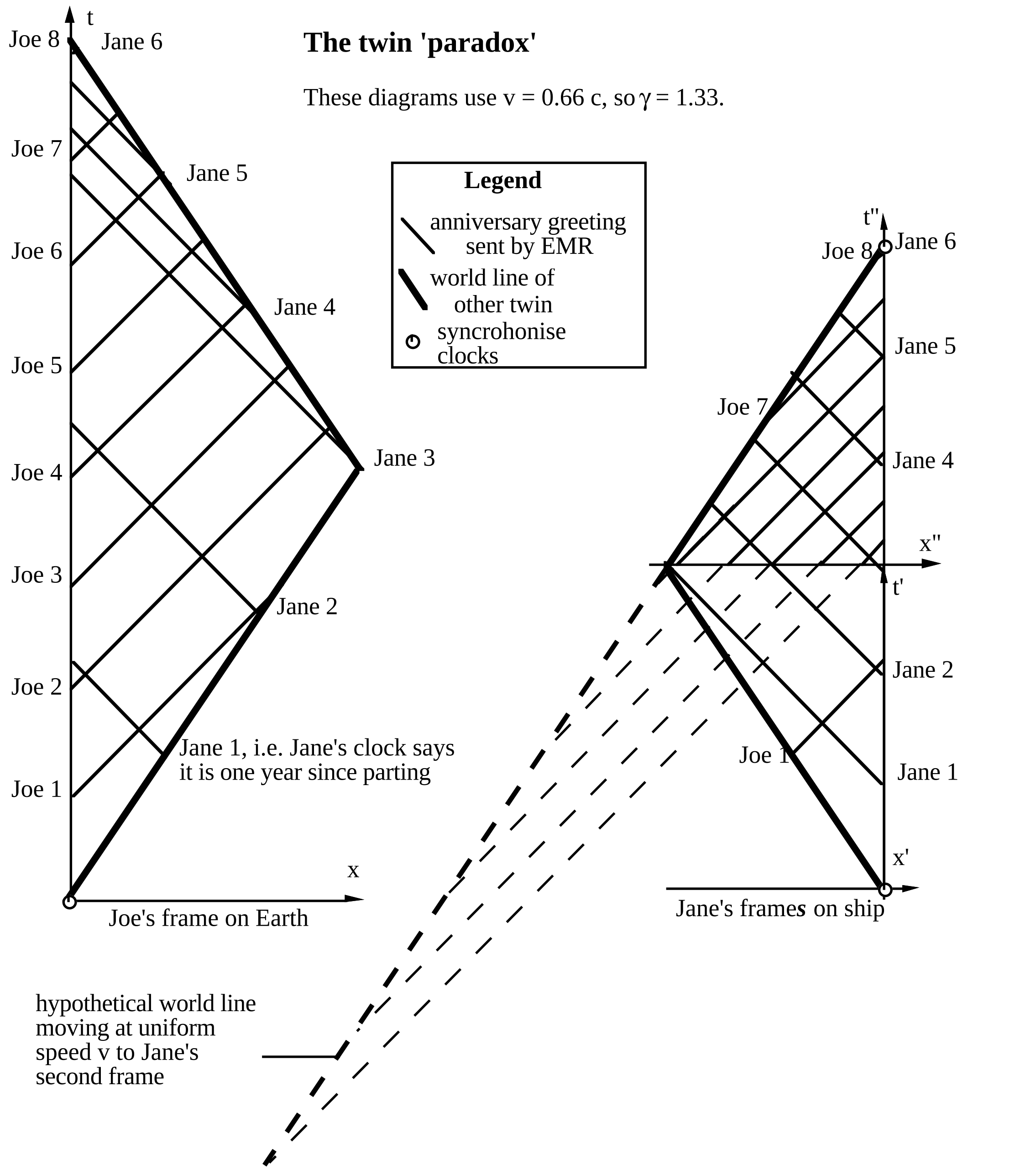
* * * * *

"Hey gramps", says Ernest, "where's my bro Algernon? He has been travelling at 0.99 c relative to me. He should now be 10/7 = 17 months old?"

* * * * *

Who is correct? http://www.phys.unsw.edu.au/~jw/twin.html

Space time diagrams *(To make the geometry easier, let's use v = 0.66 c, so γ = 1.33, and a closer turning point)*



Relativistic Mechanics

Problem. If $p_{class} \equiv mv$, momentum is only conserved in one frame. (Check using u' above.)

Define $p \equiv \gamma mv$. Check that this is conserved in both. Note that, for v << c, $p_{class} \to p$

Work Energy Theorem in Relativity

$$F = \frac{dp}{dt}$$

For force in x direction, dW = Fdx

$$= \frac{dp}{dt}dx = vdp$$

$$= v.d(\gamma mv) = mv(vd\gamma + \gamma dv)$$

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}$$

$$\therefore \quad d\gamma = -\frac{1}{2} \frac{-2v/c^2}{(1 - v^2/c^2)^{3/2}}$$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} \qquad\qquad d\gamma = \frac{\gamma^3 v}{c^2} dv$$

$$v^2 = c^2(1 - 1/\gamma^2) \qquad\qquad dv = \frac{c^2}{\gamma^3 v} d\gamma$$

$$dW = m\left(\frac{c^2}{\gamma^2} + c^2\left(1 - \frac{1}{\gamma^2}\right)\right)d\gamma = mc^2 d\gamma$$

$$K = \int_{v=0} dW = mc^2 \int_{\gamma=1} d\gamma = mc^2(\gamma - 1) \quad (*)$$

Note: as $v \to 0$,

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \to 1 + \frac{1}{2}\frac{v^2}{c^2} + ... \qquad \textit{binomial or Taylor expansion}$$

$$\therefore \; K \to \frac{1}{2}mv^2$$

Write the preceding result (*) thus:

$$\gamma mc^2 = mc^2 + K$$

We might call this $E = E_o + K$

where E_o would be like a proper energy

$\therefore \qquad E_o =$

Example What is the minimum energy released in an annihilation collision between an electron and a positron?

Minimum $2E_o = 2m_e c^2$

$= 2\,(9.11\ 10^{-31}\ \text{kg})\,(3\ 10^8\ \text{ms}^{-1})^2$

$= 1.6\ 10^{-13}\ \text{J} = 1.0\ \text{Mev}$

(proper energy of electron = 0.511 MeV)

In principle, can make electron-positron pairs with 'modest' accelerators

Example The rest energy of a proton is 938 MeV, of a neutron 940 Mev. What is the binding energy per nucleon in ^{4}He?

$$\frac{4.003}{6.02\ 10^{26}}\,c^2 = 3.735\ \text{GeV}$$

$$2m_p + 2m_n + 2m_e = 3.758\ \text{GeV}$$

Difference is ~ 20 MeV $\rightarrow$ ~ 5 MeV per nucleon

Incidentally:

data from http://physics.nist.gov/

Atomic masses:

"protium" (p,e)	$^1_1$H	1.00783
deuterium (p,n,e)	$^2_1$H	2.01410
tritium (p,2n,e)	$^3_1$H	3.01605
helium (2p,2n,2e)	$^4_2$He	4.00260

$$2m_{De} - m_{He} = 4.02820 - 4.00260$$
$$= 0.02560\ \text{au}$$

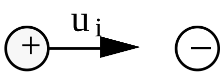
Example In the reaction

$$p + p^- \rightarrow p + p^- + p + p^-,$$

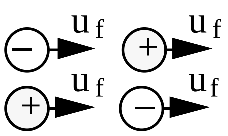
one of the reacting protons is at rest in the laboratory. What minimum accelerating voltage is required for the other?

minimum energy collision $\rightarrow$
no energy 'wasted' on motion relative to centre of mass
i.e. they al travel ~ together after the collision

Before

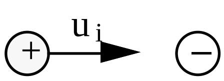


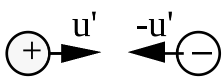
After



Let the *centre of mass* frame move at v,

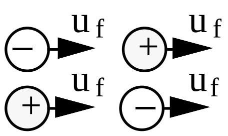
Before Lab frame C of M frame

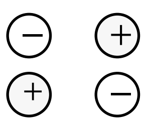




RH p$^+$ is at rest in lab, $\therefore$ v of CM frame is u'

After Lab frame C of M frame





all u'_f
small

In the CM frame $E_i \geq E_f$

$$\therefore \quad 2\gamma mc^2 \geq 4mc^2$$

$$\therefore \quad \gamma \geq 2$$

$$u' = ... = \sqrt{3/4}\,c$$

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}} = \frac{u' + u'}{1 + \frac{u'u'}{c^2}} = \frac{4\sqrt{3}}{7}\,c$$

$$qV = KE \geq \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2$$

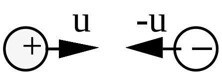
$$\Rightarrow \quad V \geq 5.6\text{ GV}$$

very wasteful experiment, so:

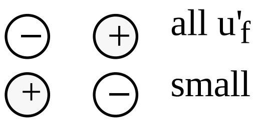
Example (As before, but): What if you collide p^+ and p^- travelling in opposite directions?

Here the lab is the C of M frame, so

Before:



After



Energy before = Energy after

$$2\gamma mc^2 = 4mc^2$$

$$\gamma = 2$$

$$u = ... = \sqrt{3/4}\,c$$

for each proton

Acceleration energy = energy to make a new proton

$$qV = mc^2$$

$$m_p = 938\text{ MeV}$$

$$\therefore \quad V = 938\text{ MV}$$

Example

'Solar constant' is 1.4 kWm^{-2}.

$r_{earth\text{-}sun} = 150\ 10^6$ km

What is the rate of mass loss of the sun due to this radiation? (i.e. neglect neutrinos, solar wind...)

$$I_{sun} = \frac{P_{sun}}{4\pi r^2} \qquad \therefore \quad P = I.4\pi r^2$$

$$\text{but} \quad P = -\frac{dE_o}{dt} = -c^2 \frac{dm}{dt}$$

$$\frac{dm}{dt} = -\frac{P}{c^2} = -\frac{I.4\pi r^2}{c^2} = \ .....$$

$= 4.4\ 10^9$ $kg.s^{-1}$

$= 1.3\ 10^{14}$ $tonnes.yr^{-1}$

$= 6\ 10^{23}$ tonnes so far

$= 1.1\ 10^{-38}$ $\%yr^{-1}$

$= 1.5\ 10^{-19}$ % so far

A useful transformation and mnemonic

$p = \gamma mv$

$\therefore \quad p^2c^2 + (mc^2)^2$

$= \gamma^2 m^2 v^2 c^2 + (mc^2)^2$

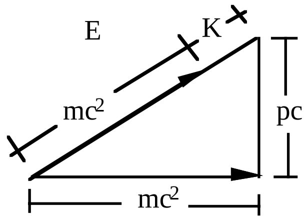
$= \dfrac{m^2v^2c^2}{1 - v^2/c^2} + (mc^2)^2$

$= \dfrac{m^2v^2c^2 + m^2c^4 - m^2v^2c^2}{1 - v^2/c^2}$

$= (\gamma mc^2)^2 = E^2$

$E = \sqrt{(pc)^2 + (mc^2)^2}$

Mnemonic:



Example What is the momentum of an electron that has been accelerated through 20.0 MV?

i) What is v?

ii) What is p/m_e?

ii) What is $p/m_e v$?

i) E_i + electrical work $= E_f$

$mc^2 + qV = \gamma mc^2$

$\therefore \quad \gamma = 1 + \dfrac{qV}{mc^2} = 1 + \dfrac{20\text{ MeV}}{0.511\text{ MeV}}$

$\therefore \quad v = \ldots. = 0.9997\ c$

ii) $E_f = 20.0\text{ MeV} + 0.511\text{ MeV}$

$E^2 = (pc)^2 + (mc^2)^2$

$p = \dfrac{\sqrt{E^2 - (mc^2)^2}}{c}$

$= \dfrac{\sqrt{20.5^2 - 0.5^2}\text{ MeV}}{c}$

$= 20.5\text{ MeV/c}$

$= 1.09\ 10^{-20}\ \text{kgms}^{-1}$

$p/m_e = 1.20\ 10^{10}\ \text{ms}^{-1}$ *units of speed*

= 40 c

iii) $p/m_e v \cong p/m_e c = 40$

Particles and antiparticles
electron plus positron

$e^- + e^+ \rightarrow$ photons

$$E_{photons} = 2\gamma m_e c^2$$

proton plus anitproton

$p^+ + p^- \rightarrow$ photons

$$E_{photons} = 2\gamma m_p c^2$$

photons $\rightarrow p^+ + p^-$?

possible but cannot control photons well enough

fast slow

$p^+ + p^- \rightarrow p^+ + p^- + p^+ + p^-$

Example. A Vogon ship is approaching Earth at 0.8 c. A Klingon ship is approaching Earth at 0.8 c from the opposite direction. To an Earth observer, it appears to have a length of L = 60 m.

i) How long does the Klingon ship appear to observers on the Vogon ship?

The proper length $L_o = \gamma L = \dfrac{L}{\sqrt{1 - \dfrac{v^2}{c^2}}} = 100$ m

How much is it contracted according to Vogons?

Need the relative velocity:

Klingons $\rightarrow u_x$ Earth $v \leftarrow$ Vogons

undashed frame dashed frame

$$u_x' = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$$

u_x' is Klingon speed wrt Vogons
u_x is Klingon speed wrt Earth
v is Vogon speed wrt Earth

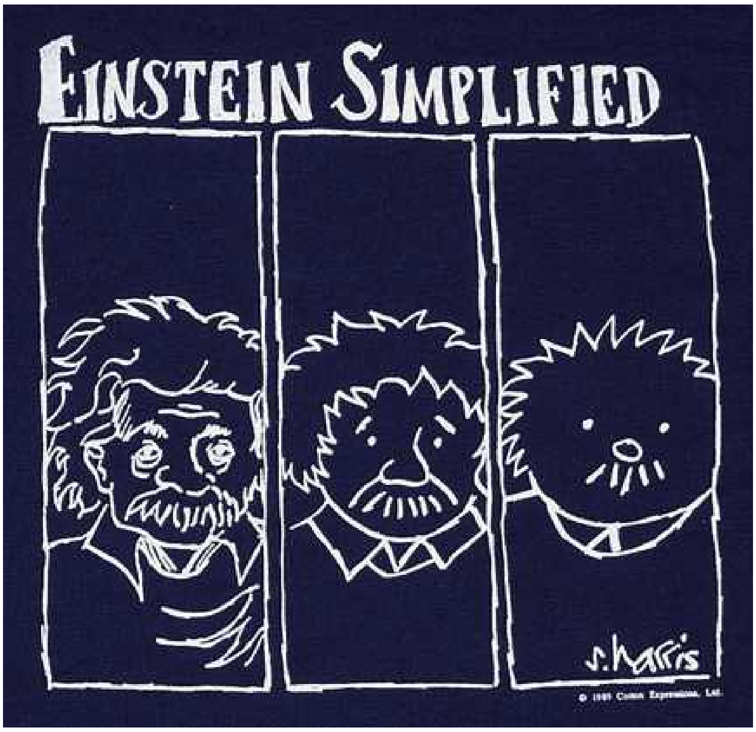
Note that v and u_x have opposite signs here.

$$u_x' = \frac{.8c - (-.8c)}{1 - \dfrac{(-.8c).8c}{c^2}} = 0.9756$$ (*carry sig figs*)

$$L' = L_o/\gamma' = \frac{100 \text{ m}}{1/\sqrt{1 - \dfrac{u'_x{}^2}{c^2}}} = 4.8 \text{ m}$$

ii) Earth observer see both ships to be one light hour away from Earth. It takes the Klingons 45 minutes to abandon ship. It takes the Vogons 30 minutes. They notice the impending collision now. Who will survive? (Think carefully.)


DAD, WILL YOU EXPLAIN THE THEORY OF RELATIVITY TO ME? I DON'T UNDERSTAND WHY TIME GOES SLOWER AT GREAT SPEED.
IT'S BECAUSE YOU KEEP CHANGING TIME ZONES. SEE, IF YOU FLY TO CALIFORNIA, YOU GAIN THREE HOURS ON A FIVE-HOUR FLIGHT, RIGHT?
SO IF YOU GO AT THE SPEED OF LIGHT, YOU GAIN MORE TIME, BECAUSE IT DOESN'T TAKE AS LONG TO GET THERE. OF COURSE, THE THEORY OF RELATIVITY ONLY WORKS IF YOU'RE GOING WEST.
GEE, THAT'S NOT WHAT MOM SAID AT ALL! SHE MUST BE TOTALLY OFF HER ROCKER.
WELL, WE MEN ARE BETTER AT ABSTRACT REASONING. GO TELL HER THAT.


EINSTEIN SIMPLIFIED
S. Harris