Circular motion

Angular position θ how far it has turned

eg: it has turned through 90°

$\theta = 90°$, $\theta = \pi/2$ radians

Angular velocity ω how fast it is turning

eg: it is turning at 3000 revolutions per minute

$$\omega = 3000 \text{ rpm} = \frac{3000 \text{ turns}}{60 \text{ s}}$$

$$= 50 \text{ turns/s} = \frac{50*2\pi}{\text{s}} = 314 \text{ rad.s}^{-1}$$

Angular acceleration α how fast ω is increasing

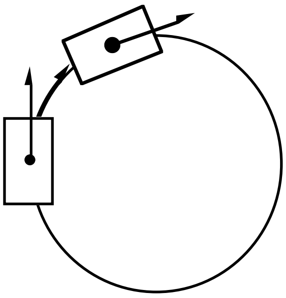
eg: it goes from 0 to 3000 r.p.m in 5 seconds

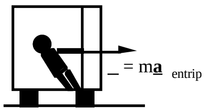
$\omega_{init} = 0$ $\omega_{final} = 314$ rad.s^{-1}

$$\text{average } \alpha \equiv \frac{\omega_{final} - \omega_{init}}{\Delta t}$$

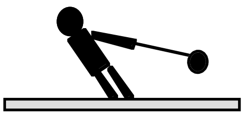
Uniform circular motion

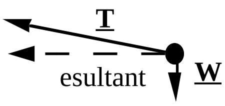
Circular motion with ω = const. Even if $\alpha = 0$, This actually produces acceleration. eg bus going round a corner





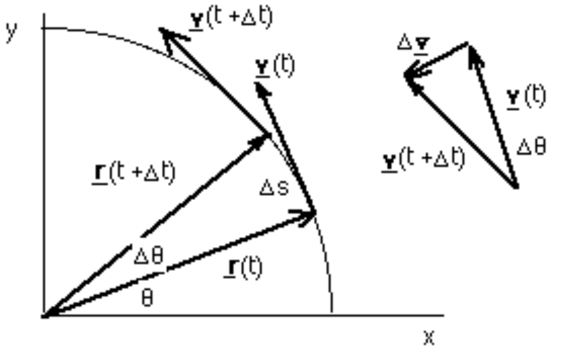
Or consider hammer thrower





Resultant force produces acceleration in the horizontal direction, towards the centre of the motion

Centripital force, centripital acceleration
Uniform circular motion



As Δt and Δθ → 0, Δ**v** → right angles to **v**

$$\therefore\ \underline{\mathbf{a}} \equiv \lim_{\Delta t\to 0}\left(\frac{\Delta \underline{\mathbf{v}}}{\Delta t}\right) \qquad \left(//\ -\underline{\mathbf{r}} \quad \text{centripital acceleration}\right)$$

As Δθ → 0, Δs → rΔθ

$$v = \frac{ds}{dt}$$

$$= r\frac{d\theta}{dt} = r\omega$$

ω is the **angular velocity**

$$|\Delta \underline{\mathbf{v}}| \cong |v\Delta\theta| \qquad (\text{n.b.: } |\Delta \underline{\mathbf{v}}| \neq \Delta|\underline{\mathbf{v}}|)$$

$$\lim_{\Delta t\to 0} \quad |d\underline{\mathbf{v}}| = |v\,d\theta|$$

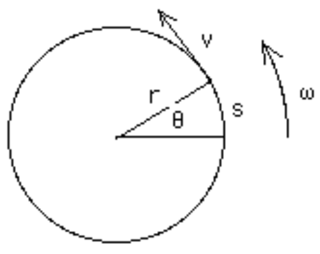
$$|\underline{\mathbf{a}}| \equiv \frac{|d\underline{\mathbf{v}}|}{dt}$$

$$= v\frac{d\theta}{dt} = v\omega$$

$$a = \frac{v^2}{r} = \omega^2 r \quad \text{but } \underline{\mathbf{a}}\ //\ -\underline{\mathbf{r}}$$

so $\underline{\mathbf{a}} = -\omega^2\underline{\mathbf{r}}$

Circular motion:



r is constant

If θ measured in radians,

$$s = r\theta.$$

$$\therefore \quad v = \frac{ds}{dt} = r\frac{d\theta}{dt} \equiv r\omega$$

$$v = r\omega \qquad \omega = \frac{v}{r}$$

$$\therefore \quad a = \frac{dv}{dt} = r\frac{d\omega}{dt} \equiv r\alpha$$

$$a = r\alpha \qquad \alpha = \frac{a}{\ }$$

Motion with constant α.

Analogies	linear	angular
displacement	x	θ
velocity	v	ω
acceleration	a	α

$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$\Delta x = v_i t + \frac{1}{2} at^2$	$\Delta\theta = \omega_i t + \frac{1}{2} \alpha t^2$
$v_f^2 = v_i^2 + 2a\Delta x$	$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$
$\Delta x = \frac{1}{2}(v_i + v_f)t$	$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$

Example. Centrifuge, initially spinning at 5000 rpm, slows uniformly to rest over 30 s. (i) What is its angular acceleration? (ii) How far does it turn while slowing down? (iii) How far does it turn during the first second of deceleration?

i) $\omega_f = \omega_i + \alpha t$ (*cf* $v_f = v_i + at$)

$= -17.5 \text{ rad.s}^{-2}$.

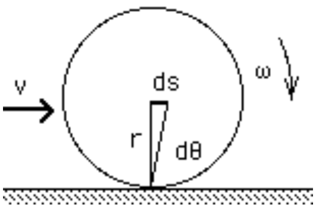
ii) $\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$ (*cf* $\Delta x = \frac{1}{2}(v_i + v_f)t$)

= 1,250 revolutions

iii) $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$ (*cf* $\Delta x = v_i t + \frac{1}{2} at^2$)

= 82 turns

Example A bicycle wheel has r = 40 cm. What is its angular velocity when the bicycle travels at 40 km.hr^{-1}?



$$v = \frac{ds}{dt}$$

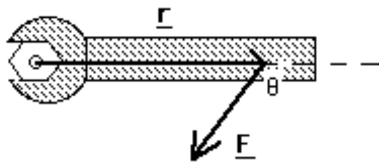
$$= \frac{r d\theta}{dt}$$

$$= r\omega$$

$$\omega = \frac{v}{r}$$

Torque

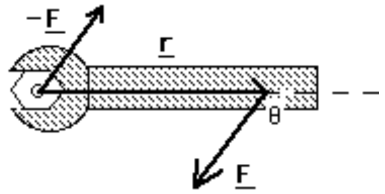
Force applied at point displaced from axis of rotation.



(Note: if **F** were only force ⇒ acceleration:

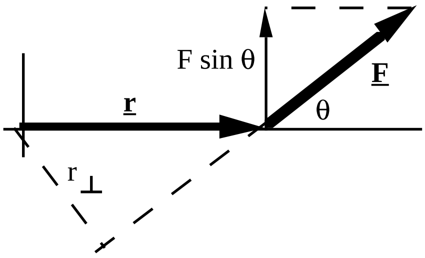
How does the 'turning tendency' depend on F? r? θ?

To get α but a = 0, need ΣF = 0.



- F does not contribute to the turning about axis.

Consider rotation about z axis



Only the component F sin θ tends to turn

$\tau = r\,(F \sin\theta)$

or $\quad = F\,(r \sin\theta) = F\, r_{\perp}$

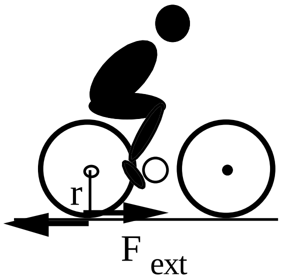
where $r_{\perp}$ is called the moment arm

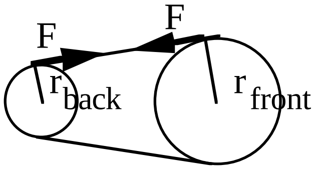
Example What is the maximum torque I apply by standing on a wheel brace 300 mm long?

$\tau = r\,(F \sin\theta)$

$\max \tau = r\,F$

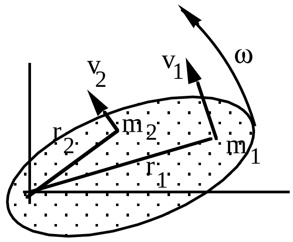
Example: bicycle and rider (m = 80 kg) accelerate at 2 ms^{-1}. Wheel with r = 40 cm. What is torque at wheel? Front sprocket has 50 teeth, rear has 25, what is torque applied by legs?





$F_{front} = F_{back}$ $\qquad \frac{r_{front}}{r_{back}} = \frac{50}{25}$

Moment of inertia *(Rotational analogue of mass)*



Choose frame so that axis of rotation is at origin

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ ....$$

$$= \frac{1}{2} m_1 (r_1\omega_1)^2 + \frac{1}{2} m_2 (r_2\omega_2)^2 + \ ....$$

$$= \frac{1}{2} \left(\Sigma\ m_i r_i^2\right) \omega^2 \qquad \textit{(cf } K = \tfrac{1}{2}mv^2\textit{)}$$

Define the **Moment of inertia**

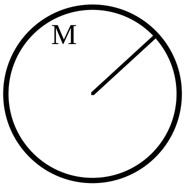
(also called moment of inertia)

System of masses $\quad I = \Sigma\ m_i r_i^2$

Continuous body $\quad I = \int_{body} r^2\, dm$

I depends on total mass, distribution of mass, shape and ***axis of rotation.*** Units are kg.m^2

Example What is I for a hoop about its axis?



All the mass is at radius r, so

$$I = Mr^2$$

For a disc: $I = \int_{body} r^2\, dm = \ ..... = \frac{1}{2} MR^2$

For a sphere $\quad I = \qquad = \frac{2}{5} MR^2$

$I = nMR^2$ $\qquad$ n is a number

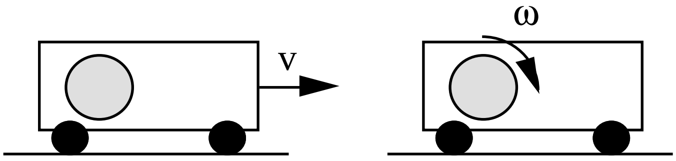
$= M\left(\sqrt{n}\,R\right)^2 = Mk^2$ $\qquad$ where $k = \sqrt{n}\,R$

$I \equiv Mk^2$ defines the **radius of gyration k**

k is the radius of a hoop with the same I as the object in question

object	I	k
hoop	MR^2	R
disc	$\frac{1}{2} MR^2$	$\frac{R}{\sqrt{2}}$
solid sphere	$\frac{2}{5} MR^2$	$\sqrt{\frac{2}{5}}\,R$

Example Use a flywheel to store the K of a bus at stops. Disc R = 80 cm, M = 1 tonne. How fast must it turn to store all the kinetic energy of a 10 t. bus at 60 $km.hr^{-1}$?



$v_m = 60\ km.hr^{-1}$ $\quad$ $v_s = 0$ *not* rolling

$\omega_m = 0$ $\quad$ $\omega_s = ?\ rev.s^{-1}$

Newton's law for rotation

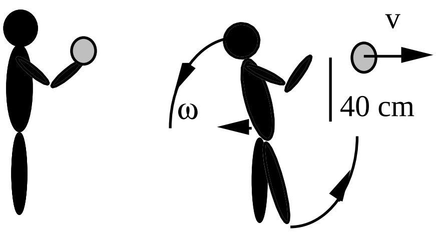
$\tau_{total} = I\alpha$ *cf* $F_{total} = ma$

Example. What constant torque would be required to stop the earth's rotation in one revolution? (Assume earth uniform.)

Plan: Know M, R, ω_i, ω_f, $\Delta\theta$. Need τ.

Use $\tau = I\alpha$, where ω_i, ω_f, $\Delta\theta \rightarrow \alpha$

Example Space-walking cosmonaut (m = 80 kg, k = 0.3 m about short axes) throws a 2 kg ball (from shoulder) at 31 ms^{-1} (**v** displaced 40 cm from c.m.). How fast does she turn? Is this a record?



In orbit so no ext torques so **L** conserved

$$\mathbf{L}_i = \mathbf{L}_f = \mathbf{L}_{ball} + \mathbf{L}_{cos}$$

Analogies: linear and rotational kinematics

Linear		**Angular**	
displacement	x	angular displacement	θ
velocity	v	angular velocity	ω
acceleration	a	angular acceleration	α

kinematic equations

$v_f = v_i + at$ $\quad$ $\omega_f = \omega_i + \alpha t$

$\Delta x = v_i t + \frac{1}{2} at^2$ $\quad$ $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$

$v_f^2 = v_i^2 + 2a\Delta x$ $\quad$ $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$

$\Delta x = \frac{1}{2}(v_i + v_f)t$ $\quad$ $\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$

Analogies: linear and rotational mechanics

mass m rotational inertia I

$I = \Sigma m_i r_i^2 \quad I = \int r^2 dm$

Work & energy

$W = \int \underline{\mathbf{F}}.d\underline{\mathbf{s}}$ $\quad W = \int \tau.d\theta$

$K = \frac{1}{2} Mv^2$ $\quad K = \frac{1}{2} I\omega^2$

force $\underline{\mathbf{F}}$ torque $\tau = rF \sin \theta$

momentum angular momentum

$\underline{\mathbf{p}} = m\, \underline{\mathbf{v}}$ $\quad |\underline{\mathbf{L}}| = mrv \sin \theta$

Newton 2:

$\underline{\mathbf{F}} = \frac{d}{dt}\underline{\mathbf{p}} = m\, \underline{\mathbf{a}}$ $\quad \underline{\boldsymbol{\tau}} = \frac{d}{dt}\underline{\mathbf{L}} = I\, \underline{\alpha}$

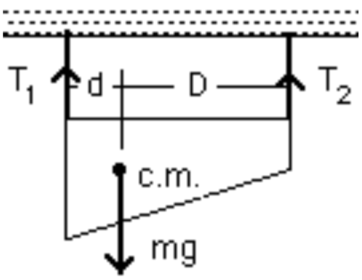
if m const if I const

Conservation of **p** and **L** :

If no external $\frac{\text{forces}}{\text{torques}}$ act on a system,

its $\frac{\text{momentum}}{\text{angular momentum}}$ is conserved.

Conservation of mechanical energy: if non-conservative forces and torques do no work, mechanical energy is conserved



Example. Object mass m suspended by two strings as shown. Find T_1 and T_2.

It's not accelerating vertically so

N2 → $\Sigma F_y = ma_y = 0$

$\therefore \quad T_1 + T_2 - mg = 0 \qquad$ (i)

It's not accelerating horizontally so

N2 → $\Sigma F_x = ma_x = 0$

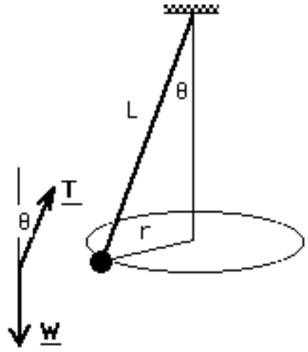
$\therefore \quad 0 = 0$ *not enough equations*

It's not **rotationally accelerating** so:

N2 → $\Sigma \tau = I\alpha = 0$

τ about c.m. clockwise $\quad \therefore \quad \tau_1 + \tau_2 = T_2 D - T_1 d = 0$

$T_1 + \frac{d}{D} T_1 - mg = 0 \rightarrow \qquad T_1 = \frac{mg}{1 + d/D} \qquad T_2 = \frac{mg}{1 + D/d}$



Example Conical pendulum (Uniform circular motion.) What is the frequency?

Apply Newton 2 in two directions:

Vertical: $a_y = 0 \quad \therefore \quad \Sigma F_y = 0$

$\therefore \quad T \cos\theta - W = 0$

$$T = \frac{mg}{\cos\theta}$$

Horizontal:

$$\frac{mv^2}{r} = a_c = T \sin\theta$$

$$= \frac{mg \sin\theta}{\cos\theta}$$

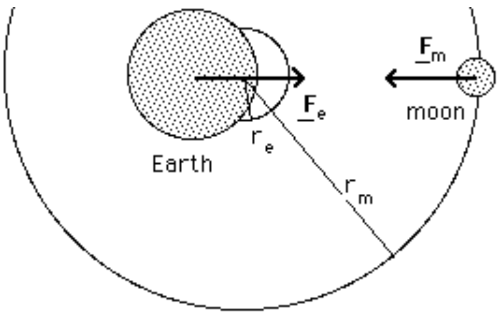
$$\therefore \quad \frac{v^2}{r} = g \tan\theta$$

$$\therefore \quad v = \sqrt{rg \tan\theta}$$

$$\therefore \quad \frac{2\pi r}{T} = \sqrt{rg \tan\theta}$$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{g \tan\theta}{r}}$$

Example Where is centre of earth-moon orbit?



$F_e = F_m = F_g$ equal and opposite
each makes a circle about common centre of mass

$F_g = m_m a_m = m_m \omega^2 r_m$

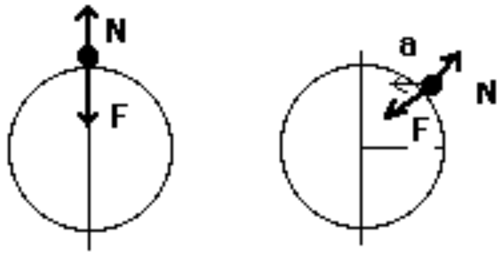
$F_g = m_e a_e = m_e \omega^2 r_e$

$$\therefore \quad \frac{r_e}{r_m} = \frac{m_m}{m_e} = \frac{5.98\ 10^{24}\ \text{kg}}{7.36\ 10^{22}\ \text{kg}} = 81.3$$

"earth-moon distance" $r_e + r_m = 3.85\ 10^8$ m

solve $\rightarrow$ $r_m = 3.80\ 10^8$ m, $r_e = 4.7\ 10^6$ m

$\therefore$ centre of orbit is inside earth



(Weight) = – (the force exerted by scales)

At poles, $\underline{\mathbf{F}} - \underline{\mathbf{N}} = 0$

At latitude θ, $\underline{\mathbf{F}} - \underline{\mathbf{N}} = m\underline{\mathbf{a}}$

where $a = r\omega^2 = (R_e \cos\theta)\omega^2$

$= .... = 0.03\ ms^{-1}$ at equator

$= 0$ at poles

We define $\underline{\mathbf{g}} = \dfrac{\underline{\mathbf{N}}}{m} = \dfrac{\underline{\mathbf{F}} - m\underline{\mathbf{a}}}{m}$

Question. How can one buy and sell gold at different latitudes so as to make a profit?

Example: In what orbit does a satellite remain above the same point on the equator?

Called the Clarke Geosynchronous Orbit

i) Period of orbit = period of earth's rotation

ii) Must be circular so that ω constant

$T = 23.9$ hours

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = ...$$

$= 44{,}000$ km popular orbit!