

# PHYS1169: Tutorial 11 Solutions

## Molecular Solids and Thermal Properties

1. In response to a temperature rise of  $\Delta T = 10K$  the volume  $V$  of mercury (approximated as  $V = 300\text{mm}^3$ ) will increase by an amount  $\Delta V$  given by

$$\Delta V = \gamma V \Delta T$$

where  $\gamma = 182 \times 10^{-6} K^{-1}$  is the thermal coefficient of volume expansion for the mercury. This increase in volume must take place in the cylindrical capillary of radius  $r = 0.05\text{mm}$ , causing the height at which the mercury sits in the capillary to change by  $\Delta h$ , where

$$\Delta V = \pi r^2 \Delta h.$$

Thus,

$$\Delta h = \frac{\Delta V}{\pi r^2} = \frac{\gamma V \Delta T}{\pi r^2} = \frac{182 \times 10^{-6} \times 300 \times 10}{\pi \times (0.05)^2} = 70\text{mm}$$

(for simplicity, all distances were expressed in millimetres.)

2. Let the original length of the steel piece be  $x_{Fe}$  and the original length of the aluminium piece be  $x_{Al}$ . These two pieces were joined to make the bar of original length  $L = 100\text{mm}$ , so

$$x_{Fe} + x_{Al} = L$$

When heated by a temperature  $\Delta T = 100K$  the composite bar increases its length by  $\Delta L = 0.218\text{mm}$ . This length increase must be due to the combination of the thermal expansions of the steel  $\Delta x_{Fe}$  and of the aluminium  $\Delta x_{Al}$ , so

$$\Delta L = \Delta x_{Fe} + \Delta x_{Al} = \alpha_{Fe} x_{Fe} \Delta T + \alpha_{Al} x_{Al} \Delta T$$

We now have two simultaneous equations for the two unknowns,  $x_{Fe}$  and  $x_{Al}$ . These can be solved using the standard techniques to give,

$$x_{Fe} = \frac{\alpha_{Al} L - \frac{\Delta L}{\Delta T}}{\alpha_{Al} - \alpha_{Fe}} \quad \text{and} \quad x_{Al} = \frac{\frac{\Delta L}{\Delta T} - \alpha_{Al} L}{\alpha_{Al} - \alpha_{Fe}}.$$

The length of the two pieces is then,

$$x_{Fe} = \frac{26 \times 10^{-6} \times 100 - \frac{0.218}{100}}{26 \times 10^{-6} - 12 \times 10^{-6}} \quad \text{and} \quad x_{Al} = \frac{\frac{0.218}{100} - 12 \times 10^{-6} \times 100}{26 \times 10^{-6} - 12 \times 10^{-6}}.$$
$$= 30\text{mm} \qquad \qquad \qquad = 70\text{mm}$$

Hence, the length of the steel section is  $30\text{mm}$  and that of the aluminium section is  $70\text{mm}$ .

3. The length change in the rod of length  $L$  when it is cooled by a temperature  $\Delta T = 55K$  is

$$\Delta L = \alpha L \Delta T$$

Young's Modulus  $Y$  is the factor that relates the stress (or pressure)  $p$  that is applied to the material to the resulting strain, or relative change in dimensions,

$$Y = \frac{p}{\Delta L/L}$$

The temperature contraction and the sheer expansion are to be the same, so

$$\Delta L = \alpha L \Delta T = \frac{pL}{Y}$$

The stress that must be applied is

$$p = \alpha \Delta T Y = 11 \times 10^{-6} \times 55 \times 120 \times 10^3 \text{ MPa} = 73 \text{ MPa}$$

4. In the thermal conduction relationship,  $H = kA \frac{dT}{dx}$ : The  $H = \frac{dQ}{dt}$ , which is the rate at which the heat flows through the area  $A$ , in that a quantity  $dQ$  (Joules) of heat energy is conducted in time  $dt$ .  $\frac{dT}{dx}$  is the temperature gradient that is causing the heat to flow, so that the temperature changes by an amount  $dT$  in a displacement  $dx$  along the path of conduction. The thermal conductivity of the medium is the constant  $k$ , which is determined by the properties of the medium through which the heat is flowing.
5. The rate of energy (heat) loss,  $H = \frac{dQ}{dt}$ , may be found using

$$H = kA \frac{dT}{dx}$$

where the quantities are as defined in question 4. The temperature gradient (assumed uniform) is the 5K drop across the 2.5mm of glass, so

$$\frac{dT}{dx} = \frac{\Delta T}{\Delta x} = \frac{5}{2.5 \times 10^{-3}} \text{ Km}^{-1}.$$

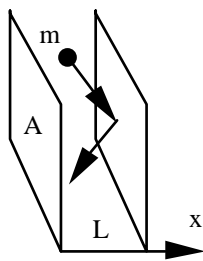
The rate of energy loss, which is simply the rate at which the heat flows out of the window,

$$H = kA \frac{dT}{dx} = 0.67 \times 2 \times \frac{5}{2.5 \times 10^{-3}} = 2.7 \text{ kW}.$$

## **Gases and Kinetic Theory**

6. Temperature is manifested in matter as the vibrations of its constituent molecules and atoms. When the temperature is high, these vibrations are very large and significant, and can contain sufficient energy to overcome any intermolecular forces that may exist between the molecules. The molecules thus have too much thermal energy to bond into relatively fixed locations relative to their neighbours, so materials tend to be gaseous at high temperatures. At low temperature, however, the thermal vibrations are small and weak, so the intermolecular forces can succeed in bonding the molecules into a regular arrangement, making a solid material.

7. Newton's second law of motion, that the force on an object is equal to the mass times the acceleration of the object or alternatively that the force is equal to the rate of change of momentum of the body, is applied to develop the kinetic theory of gases. In the kinetic theory of gases, the gas molecules move around randomly with a certain distribution of speeds. The molecules travel in a straight line unless they collide with some other object, usually either the walls of the container or other gas molecules. The pressure exerted by the gas is then interpreted as being due to the change in momentum that the gas particles experience when they collide with the walls of the container.
8. The derivation of the equation of state for an ideal gas follows the lecture notes.



parallel plates, area  $A$ . Volume  $V = AL$ .

$N$  molecules (mass  $m$ ) of an ideal gas.

Each collision  $\rightarrow$

$$|\Delta \text{ momentum}| = 2mv_x$$

The time between collisions is  $t = 2L/v_x$ .

$$\text{The average force imparted each collision} = \frac{\Delta \text{ momentum}}{\Delta \text{ time}} = \frac{2mv_x}{2L/v_x} = \frac{mv_x^2}{L}$$

$$\text{F on all } N \text{ molecules is: } F_{\text{all molecules}} = \frac{Nm \overline{v_x^2}}{L} = PA$$

Net velocity,  $v^2 = v_x^2 + v_y^2 + v_z^2$ ;

$$\text{random motion} \Rightarrow \overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} \quad \Rightarrow \quad \overline{v_x^2} = \frac{1}{3} \overline{v^2}, \text{ so:}$$

$$PAL = PV = Nm \overline{v_x^2} = \frac{N}{3} m \overline{v^2}$$

$$P = \frac{Nm}{3V} \overline{v^2} = \frac{1}{3} \rho \overline{v^2}$$

Which gives  $PV = \frac{N}{3} m \overline{v^2}$  relating the product of pressure and

volume for an ideal gas. Boltzmann's constant,  $k = 1.38 \times 10^{-23} \text{ JK}^{-1}$ , may be introduced as the universal gas constant per molecule, so the gas constant  $R$  divided by Avogadro's number  $N_A$ :

$$k = \frac{R}{N_A}$$

$$\frac{1}{2} m \overline{v^2} = \bar{\varepsilon} \equiv \text{average K.E. per molecule}$$

But T, temperature defined by:  $PV = NkT$

$$\therefore \bar{\varepsilon} = \frac{1}{2} m \overline{v^2} = \frac{3}{2} \frac{PV}{N} = \frac{3}{2} kT \quad (7)$$

For ideal gas **all energy E is kinetic** so:

$$E = N \bar{\varepsilon} = \frac{3}{2} NkT \quad (8)$$

$T \propto$  average K.E. of molecules in an ideal gas.

3 degrees of motional freedom (x, y, z)

i.e.  $\frac{1}{2} kT$  per **degree of freedom**, and we see the importance of Boltzmann's constant in that it relates the average kinetic energy of ideal gas molecules to the temperature.

9. Helium is monatomic, so there are just three degrees of freedom corresponding to the three translational directions of motion. Each degree of freedom will be associated with an average thermal energy of  $\frac{1}{2} kT$ , so the average kinetic energy of the gas molecules is

$$\bar{\varepsilon} = \frac{1}{2} m v_{rms}^2 = \frac{3}{2} kT$$

The rms speed of the gas particles is then:

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 4.216}{4.00 \times 1.67 \times 10^{-27}}} = 162 \text{ms}^{-1}$$

10. The molar mass of a molecule represent the mass in kg of one mole of the material. From the density given, we can deduce that the mass of 1 litre of the chlorine gas is 3.214 grams. If we can determine how many moles of chlorine gas make up one litre at standard temperature and pressure (STP), we could then calculate the molar mass of chlorine gas, the mass of one mole of chlorine gas. STP corresponds to a temperature  $T = 273K$  and a pressure of 1 atm, or  $P = 1.013 \times 10^5 Pa$ . Assuming chlorine may be approximated as an ideal gas, we can determine the number of moles in one litre  $V = 1L = 10^{-3} m^3$  using the ideal gas equation of state,

$$N = \frac{PV}{RT} = \frac{1.013 \times 10^5 \times 10^{-3}}{8.315 \times 273} = 0.0446 \text{mol}$$

The weight of one mole would then be

$$m_1 = \frac{3.214}{0.0446} = 72.0g$$

Therefore, the molar mass of chlorine gas is  $72.0g(mol)^{-1}$ .

### Extra Problems

E1. The molar weight of quartz ( $SiO_2$ ) is 60.1g/mol. Thus, 10 kg of quartz corresponds to

$$n = \frac{10kg}{60.1g / mol} = 166mol$$

moles of quartz.

The amount of heat energy,  $dQ$ , required to raise the temperature of  $n$  moles of material by  $dT$  is given by

$$dQ = ncdT$$

where  $c$  is the molar specific heat capacity of the material. The total heat energy required to raise the temperature of the material from 348K to 418K is

$$Q = n \int_{T=348}^{418} c dT$$

using the supplied relationship for the molar heat capacity of quartz, valid for this temperature range,

$$Q = n \int_{T=348}^{418} \left( 47 + 34 \times 10^{-3} T - \frac{11 \times 10^5}{T^2} \right) dT$$

$$Q = 166 \times \left[ 47T + 17 \times 10^{-3} T^2 + \frac{11 \times 10^5}{T} \right]_{T=348}^{418}$$

$$Q = 610kJ = 6.1 \times 10^5 J$$

E2. The thermal conductivity can be described by

$$\frac{dQ}{dt} = kA \frac{dT}{dx}$$

with the quantities as defined in earlier problems.

In the steady state, the rate of heat flow must be the power generated in the resistor,

$$\frac{dQ}{dt} = I^2 R$$

and this must be transported through every spherical shell at arbitrary radius  $r$ .

The surface area of the shell at radius  $r$  is simply  $A = 4\pi r^2$ , so the temperature gradient as a function of radius must satisfy

$$I^2 R = 4\pi kr^2 \frac{dT}{dr}$$

and so the temperature gradient cannot be constant (in fact,  $\frac{dT}{dr} \propto \frac{1}{r^2}$  to ensure the radial dependencies match on both sides).

We can integrate out the temperature gradient in the above formula, by integrating from the inner sphere (with  $r = r_1$  and  $T = T_1$ ) and the outer sphere (with  $r = r_2$  and  $T = T_2$ ),

$$\frac{dr}{r^2} = \frac{4\pi k}{I^2 R} dT$$

$$\int_{r=r_1}^{r_2} \frac{dr}{r^2} = \frac{4\pi k}{I^2 R} \int_{T=T_1}^{T_2} dT$$

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{4\pi k}{I^2 R} (T_2 - T_1)$$

and so the thermal conductivity can be found as

$$k = \frac{I^2 R (r_2 - r_1)}{4\pi r_1 r_2 (T_1 - T_2)}$$

(adjusting the sign so that the thermal conductivity is positive, while the temperature gradient is negative (as it decreases with increasing radius).

E3. Starting with the ideal gas equation of state,

$$PV = nRT$$

and making the volume the subject

$$V = \frac{nRT}{P}$$

Differentiating with respect to temperature, treating the pressure as a constant

$$\left( \frac{dV}{dT} \right)_p = \frac{nR}{P}$$

and the volume coefficient of thermal expansion is

$$\kappa = \frac{1}{V} \left( \frac{dV}{dT} \right)_p = \frac{nR}{PV} = \frac{1}{T}$$

using the equation of state once more. Therefore, at the temperature of  $T = 273.15K$  the isobaric volume coefficient of thermal expansion has the numerical value of

$$\kappa = \frac{1}{273.15} = 3.6610 \times 10^{-3} K^{-1}.$$

E4. The work done by a material as it changes its volume by  $dV$  against a pressure  $P$  is

$$dW = P dV$$

If the pressure is constant, then the work done in expanding from  $V_1$  to  $V_2$  is

$$W = P \int_{V=V_1}^{V_2} dV = P(V_2 - V_1)$$

Now, for an ideal gas, using the equation of state the work becomes

$$dW = \frac{nRT}{V} dV$$

Expanding isothermally ( $T$  is a constant) from  $V_1$  to  $V_2$ , the work done is

$$W = nRT \int_{V=V_1}^{V_2} \frac{1}{V} dV = nRT \ln\left(\frac{V_2}{V_1}\right)$$

E5. The average translational kinetic energy of each particle of an ideal gas is

$$\bar{\varepsilon} = \frac{3}{2} kT$$

One mole of the gas consist of  $N_A$  particles, so the total translational kinetic energy of the ideal gas is

$$KE = \frac{3}{2} N_A kT = \frac{3}{2} RT$$

At the temperature of  $300K$  this is

$$KE = \frac{3}{2} \times 8.315 \times 300 = 3.74kJ$$