PHYS1169: Tutorial 10 Solutions

Interference and Light Waves

1. a) $f = \frac{c}{\lambda} = \frac{3.00 \times 10^8}{589 \times 10^{-9}} = 5.09 \times 10^{14}\,Hz$.

   b) The wavelength in the medium is reduced by a factor equal to the refractive index of the medium, so $\lambda_g = \frac{\lambda}{n} = \frac{589nm}{1.52} = 388nm$.

   c) The frequency of the light will be the same in all media, so
   $$v_g = f\lambda_g = 388 \times 10^{-9} \times 5.09 \times 10^{14} = 1.97 \times 10^8.$$

2. a) The optical path length represents the equivalent distance that the wave would travel in the same time through free space. The optical path length is the length of the path taken times the refractive index in the medium, $OPL = 1.5 \times 1.6\mu m = 2.4\mu m$.

   b) $\lambda_m = \frac{\lambda}{n} = \frac{600nm}{1.5} = 400nm$.

   c) The difference in the optical path length between a wave travelling 1.6μm in free space and the same distance through the medium is $\Delta OPD = 2.4 - 1.6 = 0.8\mu m$. The phase difference between the waves $\Delta\phi$, as a fraction of 2π (one full cycle), is just the fraction of a wavelength that fits into this optical path difference, $\frac{\Delta\phi}{2\pi} = \frac{\Delta OPD}{\lambda}$. Thus, $\Delta\phi = \frac{2\pi \times 800nm}{600nm} = \frac{8\pi}{3} rad$.

3. a) For a maximum in the diffraction pattern of the parallel slits to occur at some angle, the two waves emanating from the slits must be in phase at that angular position. That is, the optical path difference between these two waves must be an integral number of wavelengths. The optical path difference can be approximated as $d\sin\theta$, where *d* is the distance between the slits and θ is the angle made by a line to the point from the perpendicular bisector of the two slits. Hence,
   $$d\sin\theta = m\lambda$$
   for constructive interference, where *m* is an integer. The central maxima is at $\theta_0 = 0$, and the first maximum is at $\theta_1 = \sin^{-1}\left(\frac{\lambda}{d}\right)$. The wavelength of the light is much smaller than the separation of the slits, $\lambda << d$ ($546nm << 0.25mm$), and so we may use the small angle approximation $\sin\theta \approx \tan\theta \approx \theta$, for θ small. If the screen is a distance *D* away, then this first bright fringe at the angle $\theta_1 = \frac{\lambda}{d}$ will appear at a distance *y* from the central maximum on the screen given by $\tan\theta_1 \approx \theta_1 = \frac{y}{D}$, so

$$y = D\theta_1 = \frac{\lambda D}{d} = \frac{546.1\times10^{-9}\times1.2}{0.25\times10^{-3}} = 2.6mm.$$

b) Minima in the diffraction pattern occur when the optical path length is an odd integer multiple of half a wavelength. Thus, $d\sin\theta = \left(m+\tfrac{1}{2}\right)\lambda$. The first minima is at $\theta_{m1} = \sin^{-1}\frac{\lambda}{2d} \approx \frac{\lambda}{2d}$, and the second is at $\theta_{m2} = \sin^{-1}\frac{3\lambda}{2d} \approx \frac{3\lambda}{2d}$. The positions of these two dark fringes on the screen are then $y_{m1} \approx \theta_{m1}D = \frac{\lambda D}{2d}$ and $y_{m2} \approx \theta_{m2}D = \frac{3\lambda D}{2d}$ respectively. The distance between the first two dark fringes is then $\Delta y = y_{m2} - y_{m1} = \frac{\lambda D}{d} = 2.6mm$.

4. Here we will consider the two primary reflections: firstly off the air-coating interface, and secondly off the coating-glass interface. We will assume that the refractive index of the coating (n = 1.38) is between that of air ($n \approx 1$) and that of glass ($n \approx 1.6$), so there will be a π radians phase change (half-cycle) on each of these reflections, as they correspond to reflection at an interface with increasing refractive index.
There will be a maximum in the reflection, corresponding to constructive interference between two reflected waves, if the optical path difference between them is an integral number of wavelengths,
$$\Delta OPD = 2nd = m\lambda$$
where *m* is an integer, since the second reflected wave must travel twice the thickness *d* of the film with refractive index *n*.
The first two wavelengths for which this condition is met are:
$$\lambda_1 = 2nd = 2\times1.38\times10^{-7} = 276nm$$
$$\lambda_2 = \frac{2nd}{2} = 138nm$$
and the other wavelengths with constructive interference will be smaller still. These wavelengths are not within the visible spectrum (300-750*nm*), so there are no reflection maxima in the visible spectrum.
In fact, the coating acts as an anti-reflective coating, as it is designed to have minimal reflection in the centre of the visible spectrum, as for destructive interference in the reflected spectrum:
$$2nd = \tfrac{1}{2}\lambda$$
which gives $\lambda = 4nd = 550nm$.

5. In this case there is a π radians phase change (half-cycle) on the reflection from the air-oil interface, but not from the oil-air interface. Thus, constructive interference will occur if the optical path difference is an integral number of wavelengths, plus one-half a wavelength, to account for this phase change on the first reflection. That is:
$$\Delta OPD = 2nd = \left(m + \tfrac{1}{2}\right)\lambda$$
so the wavelengths with constructive interference are
$$\lambda_m = \frac{2nd}{\left(m + \frac{1}{2}\right)} = \frac{2\times1.46\times500nm}{\left(m + \frac{1}{2}\right)}.$$

The first few wavelengths are (for $m = 0,1,2,3,4,5$), $2920nm$, $973nm$, $584nm$, $417nm$, $324nm$, and $265nm$. Those that lie in the visible spectrum (300-750nm) are $584nm$, $417nm$, and $324nm$, which will be wavelengths strongly reflected in the visible region.

6. Consider the interference of the wave reflected off the glass-air interface at the top of the wedge and the wave reflected off the air-glass interface at the bottom of the wedge. Constructive interference will occur if the optical path difference between these two waves is an integral number of wavelengths plus a half, to account for the half-cycle phase change (π radians) on the air-glass interface.
   Therefore, the mth bright fringe will occur at the point where the width of the wedge d_m satisfies
   $$2d_m = (m + \tfrac{1}{2})\lambda$$
   for $m = 0,1,2,\ldots$.
   The 20th bright fringe (corresponding to $m = 19$) occurs at the end of the wedge, so the thickness at this point must be
   $$d_{19} = \frac{(19 + \tfrac{1}{2}) \times 434nm}{2} = 4.2 \times 10^{-6}\, m = 4.2\mu m.$$
   This should approximately be equal to the thickness of the gap at the very end of the wedge, and so the thickness of the paper.

Diffraction and Polarisation

7. The condition for a minimum (dark fringe) in the single slit diffraction pattern is
   $$a \sin\theta = m\lambda \text{ (with } m = 1,2,3,\ldots\text{)}$$
   where a is the width of the slit, λ is the wavelength of the light, and θ is the angle at which minimum occurs, as measured from the central axis of the slit.
   The second bright fringe will be taken to be midway between the second and the third dark fringes. The second dark fringe lies at
   $$\sin\theta_2 \approx \theta_2 = \frac{2\lambda}{a} \text{ (using the small angle approximation)}$$
   and the third is at
   $$\sin\theta_3 \approx \theta_3 = \frac{3\lambda}{a}.$$
   The angular position of the 2nd bright fringe, taken to be approximately midway between them,
   $$\theta = \frac{\theta_2 + \theta_3}{2} = \frac{5\lambda}{2a}.$$
   If the screen is located a distance D away from the slit, then the position of the fringe on the screen, y from the central maxima, will satisfy
   $$\tan\theta \approx \theta = \frac{y}{D} = \frac{5\lambda}{2a}.$$
   Solving for the wavelength of the light

$$\lambda = \frac{2ay}{5D} = \frac{2 \times 0.8 \times 10^{-3} \times 1.4 \times 10^{-3}}{5 \times 0.8} = 560nm .$$

8. The minimum angular separation between two points that can be resolved by a viewing aperture *d* is

$$\theta_R = \frac{1.22\lambda}{d} .$$

a) For a 6.5 cm aperture telescope at $\lambda = 550nm$, the minimal angular resolution is $\theta_R = \frac{1.22\lambda}{d} = \frac{1.22 \times 550 \times 10^{-9}}{6.5 \times 10^{-2}} = 1.0 \times 10^{-5}\, radians$. The minimal object size *y* at a distance of $D = 1km$ is thus $y \approx \theta_R D = 1.0cm$.

b) For a human eye with aperture $d = 2.5mm$, the minimal angular resolution becomes $\theta_R = \frac{1.22\lambda}{d} = \frac{1.22 \times 550 \times 10^{-9}}{2.5 \times 10^{-3}} = 2.7 \times 10^{-4}\, radians$. The minimal object size at 1 km is then $y \approx \theta_R D = 27cm$.

9. Brewster's angle θ_B, where light reflected from the interface between two media of refractive indices n_1 and n_2 will be 100% polarised satisfies

$$\tan\theta_B = \frac{n_2}{n_1} .$$

For Brewster's angle of 48° when passing from air ($n_1 = 1.00$), the refractive index of the medium must be

$$n_2 = n_1 \tan\theta_B = 1.00 \times \tan 48 = 1.1 .$$

10. The light intensity after the first polarised sheet will be $I_1 = \frac{1}{2} I_0$, where I_0 is the intensity of the original light. This can be seen by arguing that on average half of the component of the polarised light will be aligned with the polarisation axis, and half misaligned and removed by the sheet. Alternatively and more formally, it can be seen as follows: If we define $I_{d\theta} d\theta$ to be the intensity of the light with polarisation vectors in the angular region θ to θ + *d*θ (assumed uniform, so same value for any θ), then proportion of this intensity that is transmitted will be $dI_1 = I_{d\theta} \cos^2\theta\, d\theta$. The total transmitted intensity is then $I_1 = I_{d\theta} \int_0^{2\pi} \cos^2\theta\, d\theta$, where $I_0 = I_{d\theta} \int_0^{2\pi} d\theta$. This gives the same result, that the fraction of the unpolarised light passed by the polarised sheet is $I_1 = \frac{1}{2} I_0$.

The light after the first sheet will be plane polarised, and the amount of that light that is transmitted through a polarised sheet with its axis at an angle φ to the polarisation plane of the light is

$$I_2 = I_1 \cos^2\varphi .$$

Thus, $I_2 = I_1 \cos^2\varphi = \frac{1}{2} I_0 \cos^2 30 = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 I_0 = \frac{3}{8} I_0$.

11. Here we use the relationship that the ratio of the transmitted to the incident light intensity is given by

$$\frac{I_t}{I_i} = \cos^2 \phi$$

where ϕ the angle between two polarisation axes.

a) $\phi = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 55^\circ$.

b) $\phi = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = 63^\circ$.

c) $\phi = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) = 72^\circ$.

Extra Problems

E1. The intensity as a function of the angle θ made from the central axis of the double slit interference pattern is given by

$$I(\theta) = I_m \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

where I_m is the intensity of the central maximum, λ is the wavelength of the light, and the other variables are as defined in the diagram. We can find the angle θ where the intensity will be 75% of that of the central maxima ($I = 0.75 I_m$),

$$0.75 = \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

$$\therefore \sin\theta \approx \theta = \frac{\lambda}{\pi d}\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\lambda}{6d}$$

using the small angle approximation.
The position this point makes on the screen located a distance *L* from the slits is then given by

$$\tan\theta \approx \theta = \frac{y}{L}$$

hence,

$$y = \theta L = \frac{\lambda L}{6d} = \frac{600 \times 10^{-9} \times 1.2}{6 \times 0.25 \times 10^{-2}} = 4.8 \times 10^{-5}\, m.$$

E2. a) The optical path difference between the two waves interfering at an angle of θ from the central axis of the double slits is given by

$$\Delta OPL = d \sin\theta$$

Here, we will use the small angle approximation for θ, and then expressing it in terms of the distance to the screen *L* and the distance *y* on the screen from the central maxima,

$$\sin\theta \approx \tan\theta \approx \theta = \frac{y}{L}$$

and so,

$$\Delta OPL = d\sin\theta \approx d\theta = \frac{dy}{L}$$

The phase difference $\Delta\phi$ as a fraction of a full period, which in phase is 2π, should be same as the fraction that the optical path length difference makes with a full period, which in the case of optical path length is one wavelength,

$$\frac{\Delta\phi}{2\pi} = \frac{\Delta OPL}{\lambda}$$

The phase difference between the two waves is then

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta OPL = \frac{2\pi}{\lambda}\frac{dy}{L} = \frac{2\pi \times 0.85 \times 10^{-3} \times 2.5 \times 10^{-3}}{6000 \times 10^{-10} \times 2.8} = 7.9 radians$$

b) To interfere two waves that differ in phase by $\Delta\phi$, we simply add their amplitudes. If we let the two waves be in general $E_1 = E_0 \cos(kx - \omega t)$ and $E_2 = E_0 \cos(kx - \omega t + \Delta\phi)$, then the resultant wave is then

$$E = E_1 + E_2 = 2E_0 \cos\left(\frac{\Delta\phi}{2}\right)\cos\left(kx - \omega t + \Delta\phi/2\right)$$

using the standard trigonometric result,

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

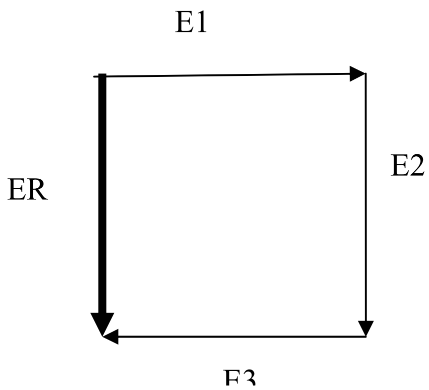
so its amplitude is $2E_0 \cos\left(\frac{\Delta\phi}{2}\right)$. Since the intensity is proportional to the square of the amplitude, the intensity of the resultant wave can be written as a function of the phase difference $\Delta\phi$ between the two interfering waves as

$$I = I_m \cos^2\left(\frac{\Delta\phi}{2}\right)$$

where I_m represents the maximum intensity attainable, if the two waves are in phase. The ratio of the intensity at this point to that of the central maximum (with maximum intensity) is then

$$\frac{I}{I_m} = \cos^2\left(\frac{\Delta\phi}{2}\right) = \cos^2\left(\frac{7.9}{2}\right) = 0.45$$

E3. The three phasors can be added like vectors, head to tail. The magnitude of each vector is just the amplitude of the corresponding field, all E_0 in this case. The angles between the vectors is just determined by the relative phase between the fields, so 0° (reference) for E_1, $3\pi/2 = 270° \equiv -90°$ for E_2, and $6\pi/2 = 540° \equiv 180°$ for E_3. The result of adding these three phasors is shown in the following diagram, with the resultant phasor being from the tail of the first to the head of the last.



It can be seen that the magnitude of the resultant is E_0, and that its phase relative to the first phasor is $3\pi/2$, so $E_R = E_0 \sin(\omega t + 3\pi/2)$.

E4. The location of a dark fringe in the single-slit diffraction pattern is given by

$$a \sin\theta = m\lambda$$

Using the small angle approximation, the location of the first and the third dark fringes on the screen will be

$$y_1 = \frac{\lambda}{a} L \text{ and } y_3 = \frac{3\lambda}{a} L$$

where L is the distance to the screen. The distance between the first and the third dark fringes is thus,

$$\Delta y = y_3 - y_1 = \frac{2\lambda}{a} L .$$

The width of the slit a is then

$$a = \frac{2\lambda L}{\Delta y} = \frac{2 \times 690 \times 10^{-9} \times 0.5}{3 \times 10^{-3}} = 0.23mm .$$

E5. The maxima in the diffraction pattern from a grating with spacing d occur at angles θ given by

$$d \sin\theta = m\lambda$$

The spacing here is $d = \frac{1}{410} mm$.

a) The two angles at which the maxima occur for λ_1 and λ_4 in the first order (m=1) spectrum are

$$\theta_1 = \sin^{-1}\left(\frac{\lambda_1}{d}\right) = \sin^{-1}\left(\frac{410.1 \times 10^{-9}}{\frac{1}{410} \times 10^{-3}}\right) = 9.68°$$

$$\theta_4 = \sin^{-1}\left(\frac{\lambda_4}{d}\right) = \sin^{-1}\left(\frac{656.3 \times 10^{-9}}{\frac{1}{410} \times 10^{-3}}\right) = 15.60°$$

so the difference between these two angles is 5.92°.

b) The two angles for λ_1 and λ_3 in the third order (m=3) spectrum are

$$\theta_1 = \sin^{-1}\left(\frac{3\lambda_1}{d}\right) = \sin^{-1}\left(\frac{3 \times 410.1 \times 10^{-9}}{\frac{1}{410} \times 10^{-3}}\right) = 30.28°$$

$$\theta_3 = \sin^{-1}\left(\frac{3\lambda_3}{d}\right) = \sin^{-1}\left(\frac{3\times 486.1\times 10^{-9}}{\frac{1}{410}\times 10^{-3}}\right) = 36.70°$$

so the angle between the two lines is 6.42°.

E6. As in the above question, the angle in the first order spectrum of the two lines is

$$\theta_1 = \sin^{-1}\left(\frac{\lambda_1}{d}\right) = \sin^{-1}\left(\frac{589.0\times 10^{-9}}{775\times 10^{-9}}\right) = 49.464°$$

$$\theta_2 = \sin^{-1}\left(\frac{\lambda_2}{d}\right) = \sin^{-1}\left(\frac{589.6\times 10^{-9}}{775\times 10^{-9}}\right) = 49.532°$$

so the angle between the two lines in the doublet in the first order spectrum is 0.0684°.