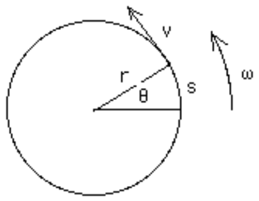
Rotational kinematics. Physics UNSW. Joe Wolfe



r is constant

If θ measured in radians,

$$s = r\theta. \qquad \textit{(definition of angle)}$$

$$\therefore \quad v = \frac{ds}{dt} = r\frac{d\theta}{dt} \equiv r\omega$$

$$v = r\omega \qquad \omega = \frac{v}{r}$$

$$\therefore \quad a = \frac{dv}{dt} = r\frac{d\omega}{dt} \equiv r\alpha$$

$$a = r\alpha \qquad \alpha = \frac{a}{r}$$

Motion with constant **$\underline{\alpha}$.**

Analogies	linear	angular
displacement	x	θ
velocity	v	ω
acceleration	a	α
	$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
	$\Delta x = v_i t + \frac{1}{2}at^2$	$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$
	$v_f^2 = v_i^2 + 2a\Delta x$	$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$
	$\Delta x = \frac{1}{2}(v_i + v_f)t$	$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$

Derivations identical - see previous. Need only remember one version

Example. Centrifuge, initially spinning at 5000 rpm, slows uniformly to rest over 30 s. (i) What is its angular acceleration? (ii) How far does it turn while slowing down? (iii) How far does it turn during the first second of deceleration?

i) $\omega_f = \omega_i + \alpha t$ (*cf* $v_f = v_i + at$)

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

$$= \frac{0 - \frac{5000*2\pi \text{ rad}}{60\text{s}}}{30\text{s}}$$

$$= -17.5 \text{ rad.s}^{-2}.$$

ii) $\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$ (*cf* $\Delta x = \frac{1}{2}(v_i + v_f)t$)

$$= \frac{1}{2}(0 + 5000\text{rpm})*0.5 \text{ min}$$

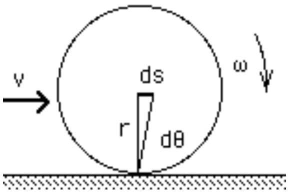
$$= 1{,}250 \text{ revolutions}$$

iii) $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$ (*cf* $\Delta x = v_i t + \frac{1}{2}at^2$)

$$= \frac{5000*2\pi \text{ rad}}{60 \text{ s}}(1 \text{ s}) - \frac{1}{2}(17.5 \text{ rad.s}^{-2}).(1 \text{ s})^2$$

$$= 515 \text{ rad} \quad (= 82 \text{ turns})$$

Example A bicycle wheel has r = 40 cm. What is its angular velocity when the bicycle travels at 40 km.hr^{-1}?



$$v = \frac{ds}{dt} = \frac{r d\theta}{dt}$$

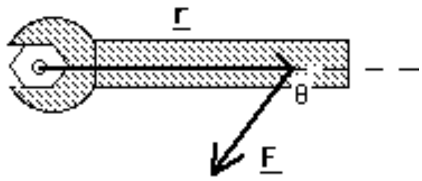
$v = r\omega$ Important result

$$\omega = \frac{v}{r} = \frac{40000 \text{ m}/3600 \text{ s}}{0.4 \text{ m}}$$

$= 28$ rad.s^{-1} (= 4.4 turns/second)

What causes angular acceleration?

Force applied at point displaced from axis of rotation.

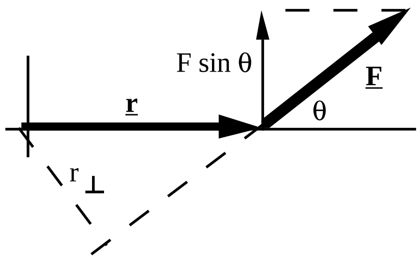


(Note: if **F** were only force ⇒ acceleration:

How does the 'turning tendency' depend on F? r? θ?

Torque. *(rotational analogue of force)*

Consider rotation about z axis



Only the component F sin θ tends to turn

$\tau = r\,(F \sin\theta)$ (*r * component of F*)

or $= F\,(r \sin\theta) = F\,r_{\perp}$ (*F * component of r*)

where $r_{\perp}$ is called the moment arm

E

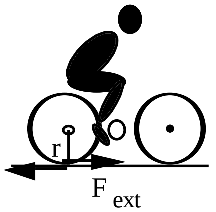
xample What is the maximum torque I apply by standing on a wheel spanner 300 mm long?

$\tau = r\,(F \sin\theta)$

$\max \tau = r F$

$= 0.3 \text{ m} * 700 \text{ N} = 200 \text{ Nm}$

if it still doesn't move: lift, use both hands or jump on it



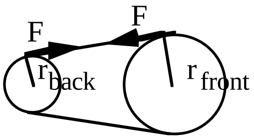
Example: bicycle and rider (m = 80 kg) accelerate at 2 ms^{-2}. Wheel with r = 40 cm. What is torque at wheel?

$F_{ext} = ma$

$\tau = rF_{ext} \sin\theta = rF$

$= rma = ... = 64$ Nm. horizontal

Front sprocket has 50 teeth, rear has 25, what is torque applied by legs?

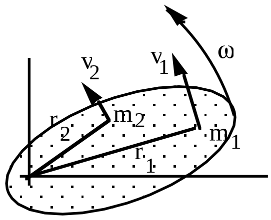


$F_{front} = F_{back}$ $\qquad \frac{r_{front}}{r_{back}} = \frac{50}{25}$

$\frac{\tau_{front}}{\tau_{back}} = \frac{r_{front}F_{front}}{r_{back}F_{back}} = 2.$

$\tau_{front} = 128$ Nm horizontal *why larger?*

Moment of inertia *(Rotational analogue of mass)*



Choose frame so that axis of rotation is at origin

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots$$

$$= \frac{1}{2} m_1 (r_1 \omega_1)^2 + \frac{1}{2} m_2 (r_2 \omega_2)^2 + \ldots$$

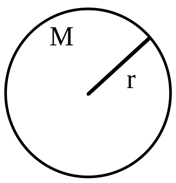
$$= \frac{1}{2} \left(\Sigma\, m_i r_i^2 \right) \omega^2 \qquad \textit{(cf } K = \tfrac{1}{2} m v^2\textit{)}$$

Define the **Moment of inertia**

System of masses $\quad I = \Sigma\, m_i r_i^2$

Continuous body $\quad I = \int_{body} r^2\, dm$

I depends on total mass, distribution of mass, shape and ***axis of rotation.*** Units are $kg.m^2$



Example What is I for a hoop about its axis?

All the mass is at radius r, so

$$I = Mr^2$$

For a disc: $I = \int_{body} r^2\, dm = \ldots\ldots = \frac{1}{2} MR^2$

For a sphere $\quad I = \qquad\qquad = \frac{2}{5} MR^2$

$I = nMR^2$ $\quad$ n is a number

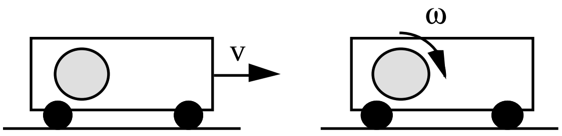
$= M\left(\sqrt{n}\,R\right)^2 = Mk^2$ $\quad$ where $k = \sqrt{n}\,R$

$I \equiv Mk^2$ defines the **radius of gyration k**

k is the radius of a hoop with the same I as the object in question

object	I	k
hoop	MR^2	R
disc	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
solid sphere	$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}\,R$

Example Use a flywheel to store the K of a bus at stops. Disc R = 80 cm, M = 1 tonne. How fast must it turn to store all the kinetic energy of a 10 t. bus at 60 $km.hr^{-1}$?



$v_m = 60\ km.hr^{-1}$ $\qquad$ $v_s = 0$ $\qquad$ *not* rolling

$\omega_m = 0$ $\qquad$ $\omega_s = ?\ rev.s^{-1}$

$$K_m = K_s$$

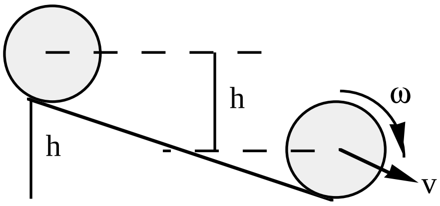
$$\frac{1}{2} M_{bus} v_m^2 = \frac{1}{2} I_{disc} \omega_s^2$$

$$M_{bus} v_m^2 = \frac{1}{2} M_{disc} R^2 \omega_s^2$$

$$\omega_s = \frac{v_m}{R} \sqrt{\frac{2M_{bus}}{M_{disc}}}$$

$$= 21\ rad.s^{-1} = 3.3\ rev.s^{-1} = 200\ rpm$$

Example. A solid sphere, a disc and a hoop roll down an inclined plane. Which travels fastest?



Rolling: point of application of friction stationary ∴ non-conservative forces do no work ∴

$$U_f + K_f = U_i + K_i$$

$$0 + \left(\frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2\right) = Mgh + 0$$

$$\omega = \frac{v}{R} \qquad \text{and write} \qquad I = Mk^2$$

$$\frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \frac{v^2}{R^2} = Mgh$$

$$\frac{1}{2} v^2 \left(1 + \frac{k^2}{R^2}\right) = gh$$

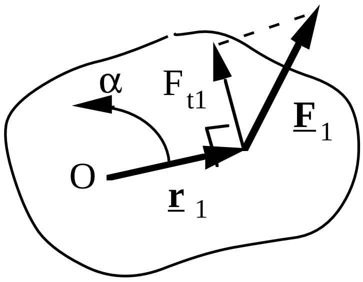
$$v = \sqrt{\frac{2gh}{1 + k^2/R^2}}$$

$$\frac{k_{sphere}}{R} = \sqrt{\frac{2}{5}} < \frac{k_{disc}}{R} = \sqrt{\frac{1}{2}} < \frac{k_{hoop}}{R} = 1$$

$\therefore \quad v_{sphere} > v_{disc} > v_{hoop}$ $\qquad$ *independent of size*

Newton's law for rotation

System of particles, m_i, all rotating with same ω and α about same axis. r_i is perpendicular distance from the axis of rotation.



$$\underline{\tau}_i \equiv \underline{r}_i \times \underline{F}_i$$
$$\tau_i = r_i F_{ti}$$

where F_t is the tangential component of F

$$\tau_i = r_i m_i a_{ti}$$
$$\tau_i = r_i m_i a_i$$
$$= r_i m_i r_i \alpha_i$$
$$\Sigma \tau_i = \Sigma m_i r_i^2 \alpha_i \qquad \text{but all } \alpha_i = \alpha$$

so $\tau_{total} = I\alpha$ and τ, α on axis

Example. What constant torque would be required to stop the earth's rotation in one revolution? (Assume earth uniform.)

Plan: Know M, R, ω_i, ω_f, $\Delta\theta$. Need τ.

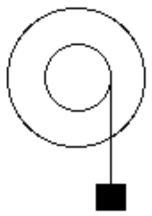
Use $\tau = I\alpha$, where ω_i, ω_f, $\Delta\theta \rightarrow \alpha$

$$\omega_f = 0, \quad \omega_i = \frac{2\pi}{23\text{h}56\text{min}} = 1.72\ 10^{-4}\ \text{rad.s}^{-1}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \qquad (cf\ \ v_f^2 = v_i^2 + 2a\Delta x)$$

$$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta}$$

$$\tau = I\alpha = \frac{2}{5} MR^2 \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta}$$
$$= \ldots$$
$$= 4.1\ 10^{28}\ \text{Nm}$$



Example Mass m on string on drum radius r on a wheel with radius of gyration k and mass M. How long does it take to turn 10 turns?

Plan: solve for a or α*, use kinematic equations.*

N2 for m (vertical): $mg - T = ma$

N2 for wheel:

$$\tau = I\alpha$$

$$rT = Mk^2.\frac{a}{r}$$

$$T = Ma\left(\frac{k}{r}\right)^2$$

$$mg - Ma\left(\frac{k}{r}\right)^2 = ma$$

$$a = \frac{mg}{m + M\left(\frac{k}{r}\right)^2}$$

$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2 \qquad \textit{cf} \quad \Delta x = v_i t + \frac{1}{2}at^2$$

$$t = \sqrt{\frac{2\Delta\theta}{\alpha}} = \sqrt{\frac{2(20\pi \text{ rad})\left(1 + \frac{M}{m}\left(\frac{k}{r}\right)^2\right)r}{g}}$$

Example A car is doing work at a rate of 20 kW and travelling at 100 kph. Wheels are r = 30 cm. What is the (total) torque applied by the drive wheels?

$P = Fv$, *so by analogy*: $P = \tau\omega$

Wheels are rolling so $\omega = \frac{v}{r}$

$$\therefore \quad \tau = \frac{P}{\omega} = \frac{Pr}{v}$$

$$= \frac{2\ 10^4 \text{ W } 0.3 \text{ m}}{10^5 \text{ m/3600 s}}$$

$$= 220 \text{ Nm}$$

(not equal to torque on tail shaft or at flywheel)

Analogies: linear and rotational kinematics

Linear		**Angular**	
displacement	x	angular displacement	θ
velocity	v	angular velocity	ω
acceleration	a	angular acceleration	α

kinematic equations

$v_f = v_i + at$ $\qquad$ $\omega_f = \omega_i + \alpha t$

$\Delta x = v_i t + \frac{1}{2} at^2$ $\qquad$ $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$

$v_f^2 = v_i^2 + 2a\Delta x$ $\qquad$ $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$

$\Delta x = \frac{1}{2}(v_i + v_f)t$ $\qquad$ $\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$

Analogies: linear and rotational mechanics

mass $\quad m$ $\qquad$ rmoment of inertia $\quad I$

$I = \Sigma m_i r_i^2 \qquad I = \int r^2 dm$

Work & energy

$W = \int \underline{\mathbf{F}}.d\underline{\mathbf{s}}$ $\qquad$ $W = \int \tau.d\theta$

$K = \frac{1}{2} Mv^2$ $\qquad$ $K = \frac{1}{2} I\omega^2$

force $\quad \underline{\mathbf{F}}$ $\qquad$ torque $\quad \underline{\boldsymbol{\tau}} \equiv \underline{\mathbf{r}} \times \underline{\mathbf{F}}$

momentum $\qquad$ angular momentum

$\underline{\mathbf{p}} = m\,\underline{\mathbf{v}}$ $\qquad$ $\underline{\mathbf{L}} = m\,\underline{\mathbf{r}} \times \underline{\mathbf{v}}$

Newton 2:

$\underline{\mathbf{F}} = \frac{d}{dt}\underline{\mathbf{p}} = m\,\underline{\mathbf{a}}$ $\qquad$ $\underline{\boldsymbol{\tau}} = \frac{d}{dt}\underline{\mathbf{L}} = I\,\underline{\alpha}$

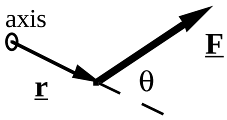
if m const $\qquad$ if I const

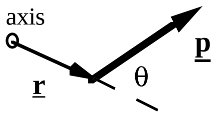
Momentum $\mathbf{p} \equiv m\mathbf{v}$

Newton 1&2 $\mathbf{F}_{ext} = \frac{d}{dt}\mathbf{p}$

Conservation law:
If no external forces act momentum conserved

If m constant, $\mathbf{F}_{ext} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a}$





$$F_{ext}\,(r \sin\theta) = \frac{d}{dt}(p\, r \sin\theta)$$

Angular momentum $L \equiv (r \sin\theta)\, mv$

only consider one axis

Newton for rotation $\tau_{ext} = \frac{d}{dt}L$

Conservation law:
If no external torques act angular momentum conserved

If I constant, $\tau_{ext} = \frac{d}{dt}L = \frac{d}{dt}I\omega = I\alpha$

I defined for a collection of particles

Conservation of **p** and **L** :
If no external $\begin{matrix}\text{forces}\\ \text{torques}\end{matrix}$ act on a system,
its $\begin{matrix}\text{momentum}\\ \text{angular momentum}\end{matrix}$ is conserved.

Conservation of mechanical energy: if non-conservative forces and torques do no work, mechanical energy is conserved

Example Particle mass m moves with

$\underline{\mathbf{r}} = (At)\,\underline{\mathbf{i}} + B\,\underline{\mathbf{j}} + \left(Ct - \frac{1}{2}gt^2\right)\underline{\mathbf{k}}$

(i) What is $\underline{\mathbf{p}}$ for the mass? (ii) What is its $\underline{\mathbf{L}}$ about the origin? (iii) what torque $\underline{\boldsymbol{\tau}}$ acts on it? (iv) What is the shape of this motion?

i) $\underline{\mathbf{p}} = m\,\underline{\mathbf{v}} = m\frac{d}{dt}\underline{\mathbf{r}}$

$= m(A\,\underline{\mathbf{i}} + (C - gt)\,\underline{\mathbf{k}})$

ii) $\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$ *recall:*

$$\begin{matrix} r_x & r_y & r_z & r_x \\ p_x & p_y & p_z & p_x \\ \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} & \underline{\mathbf{i}} \end{matrix}$$

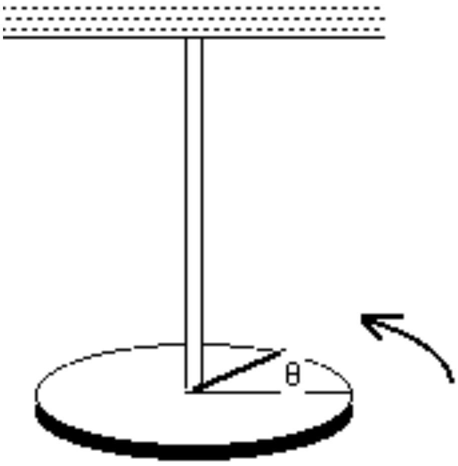
$= (r_xp_y - r_yp_x)\underline{\mathbf{k}} + (r_yp_z - r_zp_y)\underline{\mathbf{i}} + (r_zp_x - r_xp_z)\underline{\mathbf{j}}$

$\underline{\mathbf{L}} = - BmA\,\underline{\mathbf{k}} + Bm(C\text{-}gt)\underline{\mathbf{i}} +$

$\left(\left(Ct - \frac{1}{2}gt^2\right)mA - Atm(C\text{-}gt)\right)\underline{\mathbf{j}}$

$= B(C\text{-}gt)m\,\underline{\mathbf{i}} + \frac{1}{2}Amgt^2\,\underline{\mathbf{j}} - ABm\,\underline{\mathbf{k}}$

(iii) $\underline{\boldsymbol{\tau}} = \frac{d}{dt}\underline{\mathbf{L}} = - Bmg\,\underline{\mathbf{i}} + Amgt\,\underline{\mathbf{j}}$



Torsional pendulum.

Useful way of comparing unknown I with that of a simple object (e.g. rod).

Object with I is suspended on wire. The wire, when twisted, produces a restoring torque $\tau = -\kappa\theta$

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \frac{\tau}{I} = -\frac{\kappa}{I}\theta$$

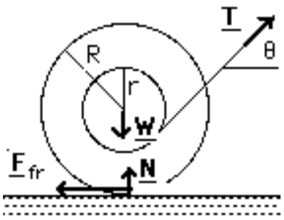
solution is:

$$\theta = \theta_m \sin(\Omega t + \phi) \quad \text{where } \Omega = \sqrt{\frac{\kappa}{I}}$$

Period

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$ *where κ is the const of the wire*

∴ for two different objects, $\frac{T_2^2}{T_1^2} = \frac{I_1}{I_2}$



Example. String round drum (r) on spool (R). What is the critical angle θ which determines direction of motion?

a) If it slides (kinetic friction $\underline{\mathbf{F}}_k$) it moves right.

b) If it rolls (static friction $\underline{\mathbf{F}}_s$) it moves left.

N2 vertical: $T \sin\theta + N = W$ (i)

N2 horizontal: $T \cos\theta = F_{fr}$ (ii)

N2 rot^n about centre $Tr = F_{fr}R$ (iii)

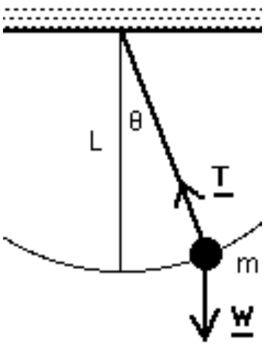
At point of sliding, $F_{fr} = \mu_s N$ (iv)

Unknowns: T, θ, F_{fr}, N. Substitute (iv):

(ii) → $T \cos\theta_c = \mu_s N$

(iii) → $Tr = \mu_s NR$

÷ → $\cos\theta_c = \frac{r}{R}$



Simple pendulum.

You may meet this in the lab.....

Mass m, suspended on light string. Radius of mass r << M ∴ treat as particle.

N2 in vertical: $mg = T \cos\theta$

N2 in horizontal: $T \sin\theta = ma = -m\frac{d^2x}{dt^2}$

If $\theta << 1$, $\sin\theta \cong \theta \cong \frac{x}{L}$, $\cos\theta \cong 1$.

$$m\frac{d^2x}{dt^2} = -T\sin\theta = -mg\frac{x}{L}$$

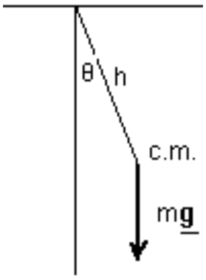
$$\frac{d^2x}{dt^2} = -T\sin\theta = -\frac{g}{L}x$$

solution is:

$$x = x_m \sin(\omega t + \phi) \quad \text{where } \omega = \sqrt{\frac{g}{L}}$$

Period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$



Physical pendulum.
Object, mass m, rotational inertia I, free to rotate.

N2 for rotation: $\tau = I\alpha$

$$-mgh\sin\theta = I\frac{d^2\theta}{dt^2}$$

If $\theta << 1$,

$$\frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\sin\theta$$

$$\frac{d^2\theta}{dt^2} \cong -\frac{mgh}{I}\theta$$

solution is:

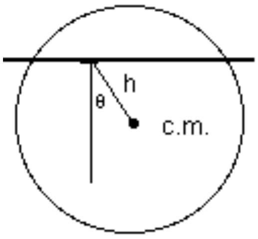
$$\theta = \theta_m \sin(\omega t + \phi) \quad \text{where } \omega = \sqrt{\frac{mgh}{I}}$$

Period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgh}}$$

(put all mass at c.m. $I = mk^2 = mh^2 \Rightarrow$ previous result)

Example.
Disc, mass m, radius R, suspended at point h from centre. What is T for this pendulum?



Period

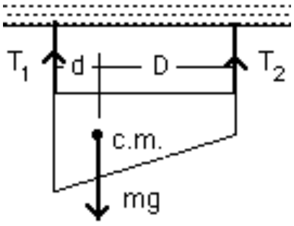
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgh}}$$

Parallel axis theorem:

$$I_{new} = I_{cm} + mh^2$$
$$= \frac{1}{2}mR^2 + mh^2$$

$$T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + h^2}{gh}}$$

(if h>> R, get $2\pi\sqrt{\frac{h}{g}}$ as for simple pendulum   if $h = 0$, $T \to \infty$)



Example. Object mass m suspended by two strings as shown. Find T_1 and T_2.

It's not accelerating vertically so

N2 → $\Sigma F_y = ma_y = 0$

$\therefore \quad T_1 + T_2 - mg = 0 \qquad$ (i)

It's not accelerating horizontally so

N2 → $\Sigma F_x = ma_x = 0$

$\therefore \quad 0 = 0$ *not enough equations*

It's not **rotationally accelerating** so:

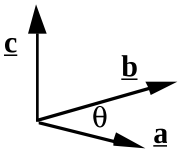
N2 → $\Sigma \tau = I\alpha = 0$

τ about c.m. clockwise $\therefore \quad \tau_1 + \tau_2 = T_2 D - T_1 d = 0$

$$T_1 + \frac{d}{D}T_1 - mg = 0$$

$$T_1 = \frac{mg}{1 + d/D} \qquad T_2 = \frac{mg}{1 + D/d}$$

The vector product (not in syllabus).



Define $|\mathbf{a} \times \mathbf{b}| \equiv ab \sin\theta$

$\mathbf{a} \times \mathbf{b}$ at right angles to $\mathbf{a}$ and $\mathbf{b}$ in right hand sense

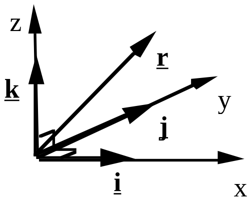
pronounced "a cross b"

For **right hand**

Thumb x **index** = **middle** (*remember TIM*)

(or **North** x **East** = **down** *remember NED*)

Turn screwdriver from $\mathbf{a}$ to $\mathbf{b}$ and (r.h.) screw moves in direction of ($\mathbf{a} \times \mathbf{b}$)



Apply to unit vectors:

$|\mathbf{i} \times \mathbf{i}| = 1 . 1 \sin 0° = 0 = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k}$

$|\mathbf{i} \times \mathbf{j}| = 1 . 1 \sin 90° = 1 = |\mathbf{j} \times \mathbf{k}| = |\mathbf{k} \times \mathbf{i}|$

$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

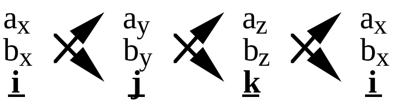
but $\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Usually evaluate by $|\mathbf{a} \times \mathbf{b}| \equiv ab \sin\theta$

but **Vector product by components** is neat

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \times (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \\ &= (a_x b_x)\, \mathbf{i} \times \mathbf{i} + (a_y b_y)\, \mathbf{j} \times \mathbf{j} + (a_z b_z)\, \mathbf{k} \times \mathbf{k} \\ &\quad + (a_x b_y)\, \mathbf{i} \times \mathbf{j} + (a_y b_z)\, \mathbf{j} \times \mathbf{k} + (a_z b_x)\, \mathbf{k} \times \mathbf{i} \\ &\quad + (a_y b_x)\, \mathbf{j} \times \mathbf{i} + (a_z b_y)\, \mathbf{k} \times \mathbf{j} + (a_x b_z)\, \mathbf{i} \times \mathbf{k}\end{aligned}$$

$$\mathbf{a} \times \mathbf{b} = (a_x b_y - a_y b_x)\mathbf{k} + (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j}$$



Example.

$\mathbf{F} = (3\,\mathbf{i} + 5\,\mathbf{j})$N, $\mathbf{r} = (4\,\mathbf{j} + 6\,\mathbf{k})$m; $\boldsymbol{\tau} = ?$

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

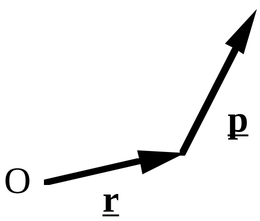
$= (r_x F_y - r_y F_x)\mathbf{k} + (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j}$

$= (0 - 4\text{ m}. 3\text{ N})\mathbf{k} + (0 - 6\text{ m}. 5\text{ N})\mathbf{i} + (6\text{ m}. 3\text{ N} - 0)\mathbf{j}$

$= \left(-30\mathbf{i} + 18\mathbf{j} - 12\mathbf{k}\right)$ Nm

Angular momentum

For a *particle* of mass m and momentum $\mathbf{p}$ at position $\mathbf{r}$ relative to origin O of an inertial reference frame, we define angular momentum (w.r.t.) O



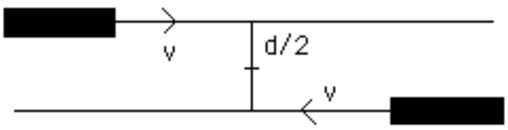
$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

or $= rp \sin\theta$

Example What is the angular momentum of the moon about the earth?

$$L = |\underline{\mathbf{r}} \times \underline{\mathbf{p}}|$$
$$= rp \sin\theta$$
$$\cong rmv \sin 90°$$
$$= mr^2\omega$$
$$= (7.4\ 10^{22}\ \text{kg})\ (3.8\ 10^{8}\ \text{m})^2 \frac{2\pi}{27.3\ 24\ 3600\ \text{s}}$$
$$= 2.8\ 10^{34}\ \text{kg m}^2\ \text{s}^{-1}$$

Direction is North



Example Two trains mass m approach at same speed v, travelling antiparallel, on tracks separated by distance d. What is their total angular momentum, as a function of separation, about a point halfway between them?

$$\underline{\mathbf{L}}_1 = \underline{\mathbf{r}}_1 \times \underline{\mathbf{p}}_1$$
$$= \underline{\mathbf{r}}_1 \times m\underline{\mathbf{v}}_1$$
$= (d/2)mv$ (clockwise on my diagram)

$\underline{\mathbf{L}}_2 = (d/2)mv$ (also clockwise)

$= dmv$ independent of separation

Newton 2 for angular momentum:

$$\Sigma\underline{\boldsymbol{\tau}} \equiv \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \underline{\mathbf{r}} \times \frac{d}{dt}\underline{\mathbf{p}}$$

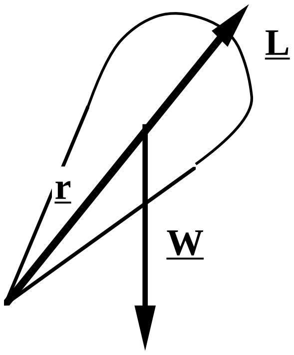
$$\frac{d}{dt}\underline{\mathbf{L}} = \frac{d}{dt}(\underline{\mathbf{r}} \times \underline{\mathbf{p}})$$

$$= \left(\frac{d}{dt}\underline{\mathbf{r}}\right) \times \underline{\mathbf{p}} + \underline{\mathbf{r}} \times \frac{d}{dt}\underline{\mathbf{p}}$$ *order important!*

$$= \underline{\mathbf{v}} \times m\underline{\mathbf{v}} + \underline{\boldsymbol{\tau}}$$

$$\Sigma\ \underline{\boldsymbol{\tau}} = \frac{d}{dt}\underline{\mathbf{L}}$$ Newton 2 in rotation

Question: A top balances on a sharp point. Why doesn't it fall over?
(Qualitative treatment only.)



$$\underline{\boldsymbol{\tau}} = \frac{d}{dt}\underline{\mathbf{L}}$$

$d\underline{\mathbf{L}} \ // \ \underline{\boldsymbol{\tau}}$

but $\underline{\boldsymbol{\tau}}$ is horizontal
so $d\underline{\mathbf{L}}$ is perpendicular to $\underline{\mathbf{g}}$

Also boomerangs, frisbees, satellites

Systems of particles

Total angular momentum $\underline{\mathbf{L}}$

$$\underline{\mathbf{L}} = \Sigma (\underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i)$$

$$\frac{d}{dt}\underline{\mathbf{L}} = \Sigma \frac{d}{dt}(\underline{\mathbf{r}}_i \times \underline{\mathbf{p}}_i)$$

$$= \Sigma\, \underline{\boldsymbol{\tau}}_i$$

$$= \Sigma\, \underline{\boldsymbol{\tau}}_{i\ \text{internal}} + \Sigma\, \underline{\boldsymbol{\tau}}_{i\ \text{external}}$$

Internal torques cancel in pairs (Newton 3)

$$\therefore \quad \Sigma\, \underline{\boldsymbol{\tau}}_{ext} = \frac{d}{dt}\underline{\mathbf{L}} \qquad cf \quad \underline{\mathbf{F}}_{ext} = \frac{d}{dt}\underline{\mathbf{P}}$$

where $\Sigma\, \underline{\boldsymbol{\tau}}_{ext}$ is the sum of all external torques.

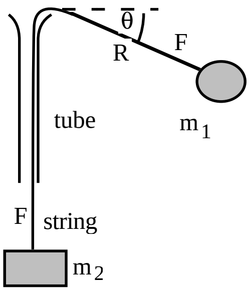
*(This equation derived for **inertial frames** but it is also true for other frames if centre of mass is taken as origin.)*

Consequence:

If $\Sigma\, \underline{\boldsymbol{\tau}}_{ext} = 0$, $\quad \frac{d}{dt}\underline{\mathbf{L}} = 0.$

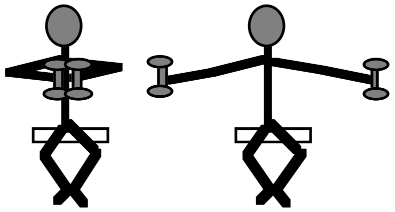
Conservation of angular momentum of isolated system

Example Circular motion of ball on string. What happens to the speed of the ball as the string is shortened? (Neglect air resistance).



Tension *does* do work, but it doesn't exert torque ($\underline{\boldsymbol{\tau}} \ // \ \underline{\mathbf{r}}$) $\therefore$ angular momentum conserved.

$$\underline{\mathbf{L}} = \underline{\mathbf{r}} \times \underline{\mathbf{p}}$$
$$= rp \sin\theta$$
$$= rmv \sin\theta$$



Example: Person on rotating seat holds two 2.2 kg masses at arms' length. Draws masses in to chest.

What is the increase in ω? Is K conserved?

Rough estimates: k_{person} about long axis ~ 15 cm

$$I_p = Mk^2 = \sim 70\ \text{kg.}\ (.15\ \text{m})^2$$ include moving part of chair

$$I_p \sim 1.6\ \text{kgm}^2$$

$$I_m = mr^2 \cong 2.2\ \text{kg.}\ (0.8\ \text{m})^2$$

$$I_m \cong 1.4\ \text{kgm}^2 \qquad \text{(arms extended)}$$

$$I_m' = mr'^2 \cong 2.2\ \text{kg.}\ (0.2\ \text{m})^2$$

$$\cong 0.1\ \text{kgm}^2 \qquad \text{(arms in)}$$

No external torques $\Rightarrow L_i = L_f$

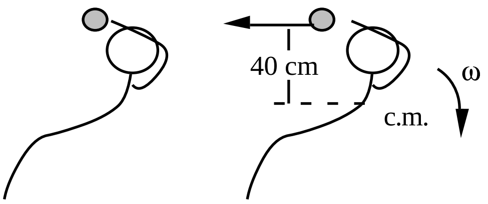
$$(I_p + 2I_m)\omega_i = (I_p + 2I'_m)\omega_f$$

$$\frac{\omega_f}{\omega_i} = \frac{I_p + 2I_m}{I_p + 2I'_m} \sim 2.4$$

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}(I_p + 2I_m)\omega_f^2}{\frac{1}{2}(I_p + 2I'_m)\omega_f^2} = 2.4$$

Arms do work: $Fds = ma_{centrip}.ds$

Example Space-walking cosmonaut (m = 80 kg, k = 0.3 m about short axes) throws a 2 kg ball (from shoulder) at 31 ms^{-1} (**v** displaced 40 cm from c.m.). How fast does she turn? Is this a record?



In orbit so no ext torques so **L** conserved

$$\mathbf{L}_i = \mathbf{L}_f = \mathbf{L}_{ball} + \mathbf{L}_{cos}$$

$$0 = \mathbf{r} \times m\mathbf{v} - I\omega$$

$$= rmv - Mk^2\omega$$

$$\omega = \frac{rmv}{Mk^2}$$

$$= 3.4\ \text{rad.s}^{-1}$$

$$= 33\tfrac{1}{3}\ \text{r.p.m.}$$