Gravity Notes for PHYS 1121-1131. Joe Wolfe, UNSW

Gravity: where does it fit in?

Gravity [general relativity]	Electric force*	Weak nuclear force	Strong nuclear force	Colour force
gravitons	photons	intermediate vector bosons	pions	gluons

electro-weak

Grand Unified Theories

Some tries for classical gravity

Theories Of Everything

* Electromagnetism "unified" by Maxwell, and also by Einstein: Magnetism can be considered as the relativistic correction to electric interactions which applies when charges move.

• Only gravity and electric force have macroscopic ("infinite") range.

$m_{graviton}? = m_{photon} = 0$

• Gravity weakest, but dominates on large scales because it is always attractive

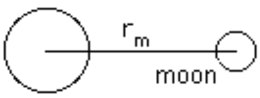
Greeks to Galileo:

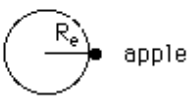
i) things fall to the ground ('natural' places)

ii) planets etc move (variety of reasons)

but no connection (in fact, natural *vs* supernatural)

Newton's calculation: accel^n of moon





$= r_m \omega_m^2$

$= (3.8\ 10^8 \text{ m})\left(\frac{2\pi}{27.3\ 24\ 3600}\right)^2$

$= 2.7\ 10^{-3} \text{ m.s}^{-2}$

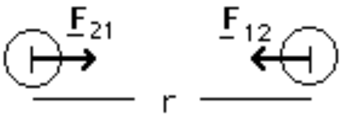
accel^n of "apple" = 9.8 ms^{-2}

$\frac{a_{apple}}{a_{moon}} = 3600; \quad \frac{r_m}{R_e} = \frac{385000 \text{ km}}{6370\text{km}} = 60;$

$\left(\frac{r_m}{R_e}\right)^2 = 3600$

Newton's brilliant idea: What if the apple and the moon accelerate according to the same law? →
What if every body in the universe attracts every other, inverse square law?

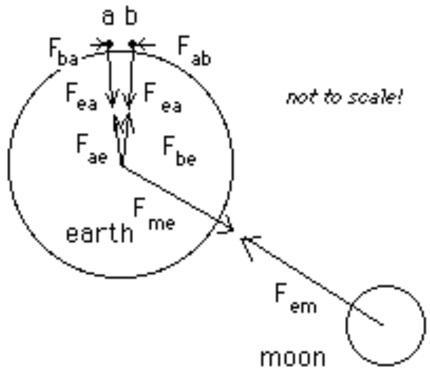
Newton's law of gravity:



$$F = -G\frac{m_1 m_2}{r^2}$$ Negative sign means **F** // − **r**

Why is it inverse square? Wait for Gauss' law in electricity.

$\underline{\mathbf{F}}_{12} = -\underline{\mathbf{F}}_{21}$ *Newton 3*



Newton already knew Kepler's empirical law:

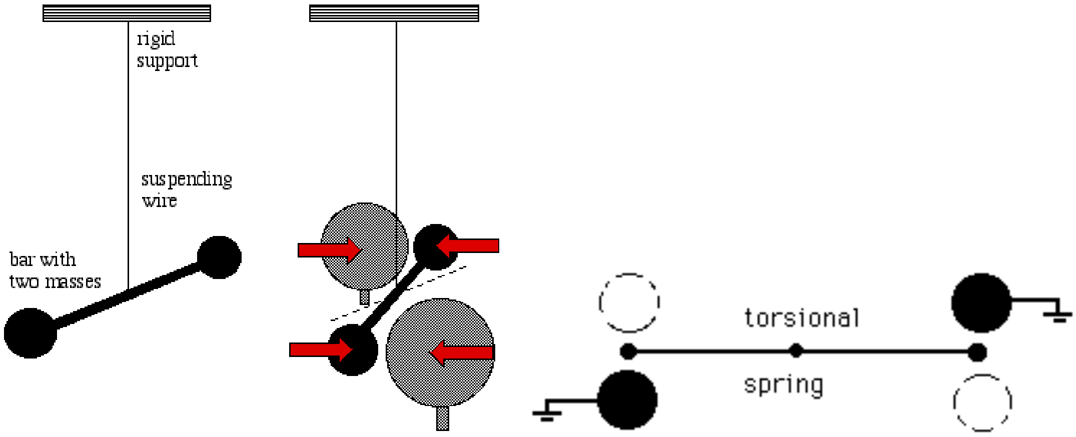
For planets, $r^3 \propto T^2$ orbit radius and period

Now if $F \propto a_c \propto \frac{1}{r^2}$

then constant = $a_c r^2 = r\omega^2 r^2 = r^3\omega^2$

Planet	r from sun	T	ω	$r\omega^2$	$r^3\omega^2$
	million km	Ms	rad.s^{-1}	ms^{-2}	
Mercury	58	7.62	8.25 10^{-7}	3.95 10^{-5}	1.31 10^{20} m^{3}s^{-2}
Venus	108	19.4	3.23 10^{-7}	1.13 10^{-5}	1.32 10^{20} m^{3}s^{-2}
Earth	150	31.6	1.99 10^{-7}	5.94 10^{-6}	1.33 10^{20} m^{3}s^{-2}
etc					

How big is G? **Cavendish's experiment** (1798)



$$F = -G \frac{m_1 m_2}{r^2}$$

From deflection and spring constant, calculate F, know m_1 and m_2, ∴ can calculate G. $G = 6.67\ 10^{-11}\ Nm^2kg^{-2}$ (*or* $m^3kg^{-1}s^{-2}$

Now also weight of m: $|W| = mg \cong G \frac{m.M_e}{R_e{}^2}$

∴ Cavendish first calculated mass of the earth:

$$M_e = \frac{gr_e{}^2}{G} = \frac{9.8\ m.s^{-2} \times (6.37\ 10^6\ m)^2}{6.67\ 10^{-11}\ Nm^2kg^{-2}} = 6.0\ 10^{24}\ kg$$

see http://www.physicscentral.com/action/action-01-5-print.html
and http://physics.usask.ca/~kolb/p404/cavendish/

Some numbers

What is force between two oil tankers at 100 m?

$$F = -G \frac{m_1 m_2}{r^2}$$

What happens when more there are ≥ 3 bodies?

Superposition principle.

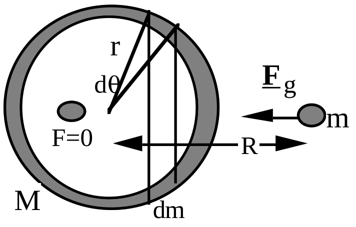
$$\underline{\mathbf{F}}_{\text{all objects together}} = \Sigma\, \underline{\mathbf{F}}_{\text{individual}}$$

or $\underline{\mathbf{F}}_1 = \sum_i \underline{\mathbf{F}}_{1i}$ force on m_1 due to masses m_i

continuous body $\underline{\mathbf{F}}_1 = \int_{\text{body}} d\,\underline{\mathbf{F}}$

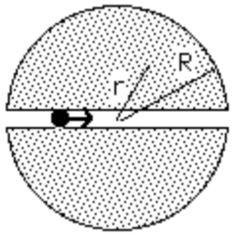
Shell theorem

A uniform shell of mass M causes the same gravitational force on a body outside is as does a point mass M located at the centre of the shell, and zero force on a body inside it.



$$\underline{F}_g = \frac{GMm}{R^2}$$

Example. If ρ_{earth} were uniform (it isn't), how long would it take for a mass to fall through a hole through the earth to the other side?



$$M_r = \rho.\frac{4}{3}\pi r^3$$

$$\therefore\ F_r = -G\frac{m\rho.\frac{4}{3}\pi r^3}{r^2}$$

$$F = -Kr \quad \text{where } K = Gm\rho.\frac{4}{3}\pi$$

$\therefore$ motion is SHM with $\omega = \sqrt{\dfrac{K}{m}}$ Simple Harmonic Motion: discussed later

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{G\rho.\frac{4}{3}\pi}} = \frac{2\pi}{\sqrt{GM/R^3}} = ... = 84 \text{ minutes}$$

$\therefore$ falls through (one half cycle) in 42 minutes *(actually faster for real density profile)*

Gravity near Earth's surface

$$W = |F_g| = G\frac{Mm}{R_e^2}$$

$$W = mg_o = G\frac{M_e m}{r^2}$$

g_o is accel^n in an inertial (non-rotating) frame

$$g_o = G\frac{M_e}{r^2}$$

Usually, $r \cong R_e$, but

$$g_o = G\frac{M_e}{(R_e + h)^2} = g_s\left(\frac{R_e}{R_e + h}\right)^2$$

$$= g_s\left(\frac{1}{1 + h/R_e}\right)^2$$

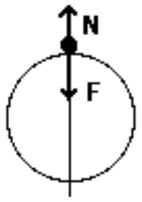
where g_s is g_o at surface

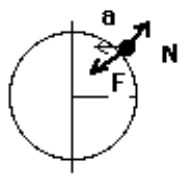
Other complications:

i) Earth is not uniform (especially the crust) *useful for prospecting*

ii) Earth is not spherical

iii) Earth rotates (*see Foucault pendulum*)





(Weight) = – (the force exerted by scales)

At poles, $\underline{\mathbf{F}} - \underline{\mathbf{N}} = 0$

At latitude θ, $\underline{\mathbf{F}} - \underline{\mathbf{N}} = m\underline{\mathbf{a}}$

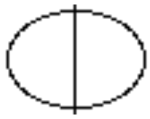
where $a = r\omega^2 = (R_e \cos\theta)\omega^2$

$= .... = 0.03\ \mathrm{ms}^{-2}$ at equator

$= 0$ at poles

We define $-\underline{\mathbf{g}} = \dfrac{\underline{\mathbf{N}}}{m} = \dfrac{\underline{\mathbf{F}} - m\underline{\mathbf{a}}}{m}$

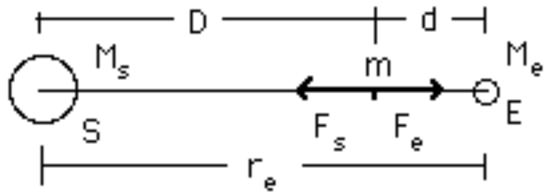
So $\underline{\mathbf{g}}$ is greatest at the poles, least at the equator, and does not (quite) point towards centre.



horizontal $\perp$ $\underline{\mathbf{g}}$

Earth is flattened at poles

Puzzle: How far from the earth is the point at which the gravitational attractions towards the earth and that towards the sun are equal and opposite? Compare with distance earth-moon (380,000 km)



$$|F_e| = |F_s|$$

$$\frac{GM_e m}{d^2} = \frac{GM_s m}{(r_e - d)^2}$$

$$M_e(r_e - d)^2 = M_s d^2$$

$$r_e^2 - 2r_e d + d^2 = \frac{M_s}{M_e} d^2$$

$$\left(\frac{M_s}{M_e} - 1\right) d^2 + 2r_e d - r_e^2 = 0$$

$$d = ... = ?$$

Gravitational field. A field is ratio of force on a particle to some property of the particle. For gravity, (gravitational) mass is the property:

$$\frac{\underline{\mathbf{F}}_{grav}}{m} = \underline{\mathbf{g}} = \underline{\mathbf{g}}(\underline{\mathbf{r}}) \qquad \text{is a vector field}$$

cf electric field $\frac{\underline{\mathbf{F}}_{elec}}{q} = \underline{\mathbf{E}}(\underline{\mathbf{r}})$ *later in syllabus*

Gravitational potential energy. Revision:

Potential energy

For a **conservative** force $\underline{\mathbf{F}}$ (i.e. one where work done against it, W = W($\underline{\mathbf{r}}$)) we can define potential energy U by $\Delta U = W_{against}$. i.e.

$$\Delta U = -\int_i^f \underline{\mathbf{F}} \cdot \underline{\mathbf{dr}}$$

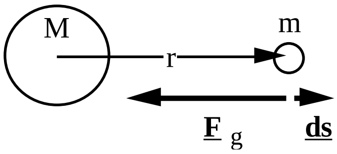
near Earth's surface, $\underline{\mathbf{F}}_g = m\mathbf{g} \cong$ constant

$$= -\int_i^f (-mg\underline{\mathbf{k}}) \cdot (dx\underline{\mathbf{i}} + dy\underline{\mathbf{j}} + dz\underline{\mathbf{k}})$$

$$= mg\,\underline{\mathbf{k}} \cdot \underline{\mathbf{k}} \int_i^f dz \qquad = mg\,(z_f - z_i)$$

choose reference at $z_i = 0$, so $U = mgz$

Gravitational potential energy of m and M.



$$\Delta U_g \equiv -\int_i^f \underline{\mathbf{F}}_g \cdot \underline{\mathbf{ds}} = \int_i^f F_g dr = \int_i^f G\frac{Mm}{r^2} dr = -GMm\left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

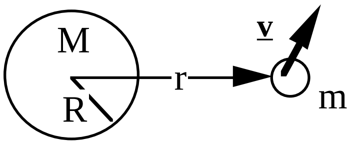
Convention: take $r_i = \infty$ as reference: $U(r) = -\frac{GMm}{r}$

U = work to move one mass from ∞ to r in the field of the other. Always negative.

Usually one mass >> other, we talk of U of one in the field of the other, but it is U of both.

Escape "velocity".

Escape "velocity" is **minimum** speed v_e required to escape, i.e. to get to a large distance ($r \to \infty$).



Projectile in space: no non-conservative forces so conservation of mechancial energy

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m v_e^2 - \frac{GMm}{R} = 0 + 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

For Earth: $$v_{esc} = \sqrt{\frac{2\ 6.67\ 10^{-11}\ m^3kg^{-1}s^{-2}\ 5.98\ 10^{24}\ kg}{6.37\ 10^6\ m}}$$

$$= 11.2\ km.s^{-1} = 40{,}000\ k.p.h.$$

Put launch sites near equator: $v_{eq} = R_e\omega_e = 0.47\ km.s^{-1}$

Question In Jules Verne's "From the Earth to the Moon", the heros' spaceship is fired from a cannon*. If the barrel were 100 m long, what would be the average acceleration in the barrel?

$$v_f^2 - v_i^2 = 2as$$

$$a = \frac{v_e^2 - 0}{2s} = \frac{(1.12\ 10^4\ ms^{-2})^2}{2 \times 100\ m}$$

$$= 630{,}000\ ms^{-2} = 64{,}000\ g$$

* why? If you burn all the fuel on the ground, you don't have to accelerate and to lift it. *Much* more efficient.

Planetary motion

"The music of the spheres" - Plato

Leucippus & Democritus C5 B.C.

heliocentric universe

Hipparchus (C2 BC) & Ptolemy (C2 AD) geocentric universe

Tycho Brahe (1546-1601) - very many, very careful, naked eye observations.

Johannes Kepler joined him. He fitted the data to these *empirical* laws:

Kepler's laws:

1 All planets move in elliptical orbits, with the sun at one focus.

Except for Pluto and Oort cloud objects, these ellipses are ≅ circles.

$M_{sun} >> m_{planet}$, *so sun is ≅ c.m.*

2 A line joining the planet to the sun sweeps out equal areas in equal time.

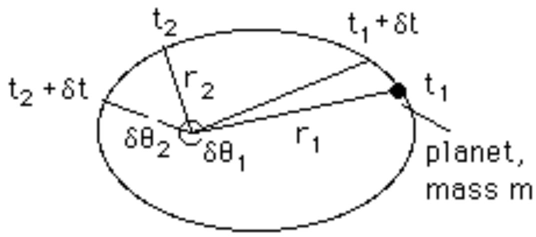
Slow at apogee (distant), fast at perigee (close)

3 The square of the period ∝ the cube of the semi-major axis

Slow for distant, fast for close

Newton's explanations:

Law of areas:



Area $= \frac{1}{2}$ r.rδθ

i.e. for same δt, $\frac{1}{2} r^2 \delta\theta$ = constant

Conservation of angular momentum **L**. Sun at c.m.

$\therefore \quad |\underline{\mathbf{L}}| = |\underline{\mathbf{r}} \times \underline{\mathbf{p}}| = |\underline{\mathbf{r}} \times m\underline{\mathbf{v}}|$ *momentum = **p**. see later*

$= mrv_{tangential}$

$= mr.r\omega = mr^2\frac{\delta\theta}{\delta t}$

$= \frac{m}{\delta t} r^2 \delta\theta$ = constant.

Conservation of **L** $\Rightarrow$ Kepler 2.

Law of periods: *(we consider only circular orbits)*

Kepler 3: $T^2 \propto r^3$

Newton 2 $\rightarrow$ $F = ma = m\, r\omega^2$

$$G\frac{Mm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \quad \rightarrow \text{ Kepler 3}$$

(works for elipses with semi-major axis a instead of r)

Newton 2 & Newton's gravity $\Rightarrow$ Kepler 3

Newton 2 & Newton's gravity also $\Rightarrow$ Kepler 1

Example What is the period of the smallest earth orbit? ($r \cong R_e$)
What is period of the moon? ($r_{moon} = 3.82\ 10^8$ m)

$$T_1 = \sqrt{\left(\frac{4\pi^2}{GM}\right) r^3} = \sqrt{\frac{4\ \pi^2}{6.67\ 10^{-11}\ 5.98\ 10^{24}} (6.37\ 10^6)^3}\ \text{s}$$

$= 84$ min

Kepler 3: $T^2 \propto r^3$

$$\frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} = \left(\frac{3.82\ 10^8}{6.37\ 10^6}\right)^{3/2} = 464$$

$T_2 = 464\ T_1 = 27.2$ days

For other planets: most have moons, so the mass of the planet can be calculated from

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

Orbits and energy

No non-conservative forces do work, so mechanical energy is constant:

$$E = K + U$$
$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Let's remove v. Consider circular orbit:

$$\frac{v^2}{r} = a_c = \frac{F}{m} = \frac{GMm}{r^2 m}$$

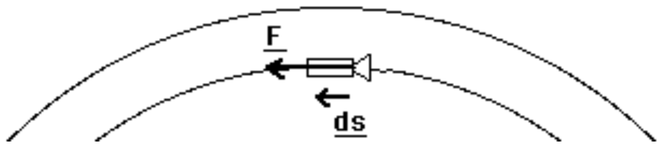
$$\therefore \quad \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$$

$$E = K + U$$
$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$
$$= -\frac{GMm}{2r}$$

i.e. $E = \frac{1}{2}U$, or $K = -\frac{1}{2}U$, or $K = -E$.

Small r $\Rightarrow$ U very negative, K large. *(inner planets fast, outer slow)*

Example A spacecraft in orbit fires rockets while pointing forward. Is its new orbit faster or slower?



F // **ds** $\therefore$ Work done on craft

$$W = \int \mathbf{F} \cdot \mathbf{ds} > 0.$$

$\therefore \quad E = -\frac{GMm}{2r}$ increases, i.e. it becomes less negative. (R is larger). K = - E, $\therefore$ K smaller, so it travels *more slowly*.

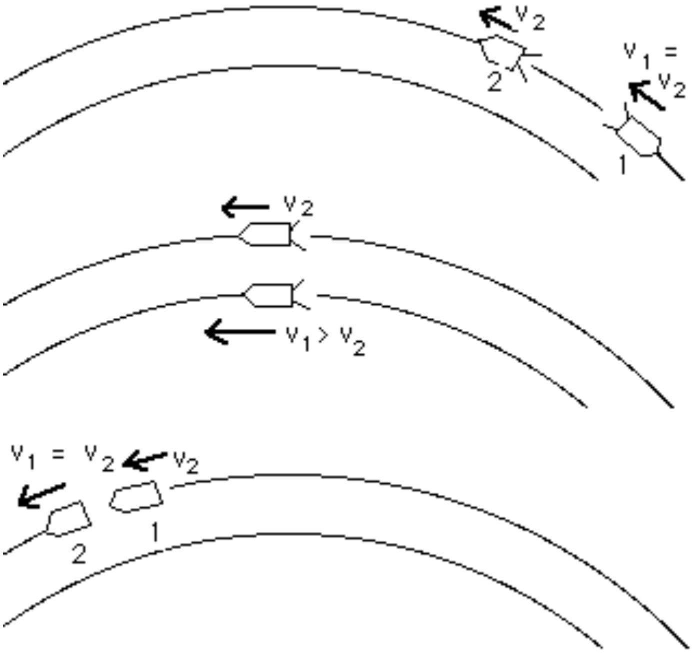
called "Speeding down"

Quantitatively:

$$K_i = -E_i \quad K_f = -E_f = -(E_i + \Delta E)$$
$$K_f = K_i - \Delta E$$
$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_f^2 - W$$

Looks odd, but need lots of work to get to a high, slow orbit.

Manœuvring in orbit.



To catch up, vessel 1 fires engines *backwards*, and loses energy. It thus falls to a lower orbit where it travels faster, until it catches up. It then fires its engines *forwards* in order to slow down (it climbs back to the original, slower orbit).

Example: In what orbit does a satellite remain above the same point on the equator?

Called the Clarke Geosynchronous Orbit

i) Period of orbit = period of earth's rotation

ii) Must be circular so that ω constant

$$T = 23.9 \text{ hours}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \;.....$$

$$= 42{,}000 \text{ km}$$ popular orbit!