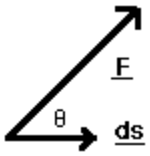
PHYS 1A Work and Energy   Joe Wolfe, UNSW

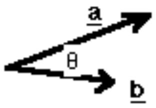
The scalar product.   *dot product*

Why? e.g. Work: scalar, related to **F**, **ds** and θ.



$dW = |\underline{\mathbf{F}}|\,|\underline{\mathbf{ds}}| \cos\theta$

(also $dV = |\underline{\mathbf{E}}|\,|\underline{\mathbf{ds}}| \cos\theta$ etc)



∴ define

$\underline{\mathbf{a}}.\underline{\mathbf{b}} \equiv ab\cos\theta \quad (= \underline{\mathbf{b}}.\underline{\mathbf{a}})$

Apply to unit vectors:

$\underline{\mathbf{i}}.\underline{\mathbf{i}} = 1\,.\,1\cos 0° = 1 = \underline{\mathbf{j}}.\underline{\mathbf{j}} = \underline{\mathbf{k}}.\underline{\mathbf{k}}$

$\underline{\mathbf{i}}.\underline{\mathbf{j}} = 1\,.\,1\cos 90° = 0 = \underline{\mathbf{j}}.\underline{\mathbf{k}} = \underline{\mathbf{k}}.\underline{\mathbf{i}}$

Scalar product by components

$$\begin{aligned}\underline{\mathbf{a}}\,.\,\underline{\mathbf{b}} &= (a_x\,\underline{\mathbf{i}} + a_y\,\underline{\mathbf{j}} + a_z\,\underline{\mathbf{k}}).(b_x\,\underline{\mathbf{i}} + b_y\,\underline{\mathbf{j}} + b_z\,\underline{\mathbf{k}})\\ &= (a_xb_x)\,\underline{\mathbf{i}}\,.\underline{\mathbf{i}} + (a_yb_y)\,\underline{\mathbf{j}}.\underline{\mathbf{j}} + (a_zb_z)\,\underline{\mathbf{k}}.\underline{\mathbf{k}}\\ &\quad + (a_xb_y + a_yb_x)\,\underline{\mathbf{i}}.\underline{\mathbf{j}} + (..)\,\underline{\mathbf{j}}.\underline{\mathbf{k}} + (..)\,\underline{\mathbf{k}}.\underline{\mathbf{i}}\end{aligned}$$

$$\underline{\mathbf{a}}\,.\,\underline{\mathbf{b}} = a_xb_x + a_yb_y + a_zb_z$$

Problem. Find the angle between

$\underline{\mathbf{a}} = 4\,\underline{\mathbf{i}} - 3\,\underline{\mathbf{j}} + 7\,\underline{\mathbf{k}}$

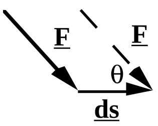
$\underline{\mathbf{b}} = 2\,\underline{\mathbf{i}} + 5\,\underline{\mathbf{j}} - 3\,\underline{\mathbf{k}}$

$ab\cos\theta = \underline{\mathbf{a}}\,.\,\underline{\mathbf{b}} = a_xb_x + a_yb_y + a_zb_z$

$$\cos\theta = \frac{a_xb_x + a_yb_y + a_zb_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\,\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

$\rightarrow \theta = 122°$

Definition of work



When force varies, use differential displacement **ds**

$dW = F\,ds\cos\theta$

(F) (ds cosθ) → F X component of ds // F, or

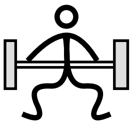
(F cosθ) (ds) → ds X component of F // ds

$$W = \int_0^L F\cos\theta\,ds$$

if F and θ are constant, we get $W = FL\cos\theta$

SI Unit: 1 Newton x 1 metre = 1 Joule

Example. How much work is done by lifting 100 kg vertically by 1.8 m very slowly?

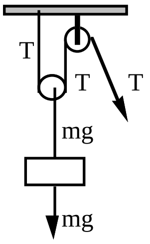


Slow $\therefore$ $F_{applied} \cong mg$

W = mg d cos 0°

= 1.8 kJ.

Not a lot, yet it is hard to do, because the force is inconveniently large. Consider:



If the rope and pulleys are light, and if the acclerations are negligible, then

Force on LH pulley

$ma \cong 0 = 2T - mg$

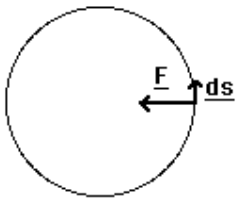
$\therefore$ $T = mg/2$

If mass rises by D, word done = mgD.

But rope shortens on both sides of rising pulley,

if mass rises by D, rope must be pulled 2D,

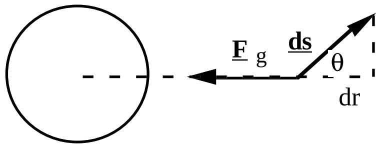
so work done = T * 2D = mgD



Example. What is the work done by gravity in a circular orbit?

$$W = \int F\, ds \cos\theta = 0$$

Example. $F_{grav} \propto 1/r^2$. How much work is done to move m = 1 tonne from earth's surface (r = 6500 km) to r = ∞?



$$W = \int F\, ds \cos\theta$$

$$= \int F\, dr$$

$$F = -F_{grav} = \frac{Cm}{r^2}$$

more later

On surface $F/m = 9.8\ ms^{-2}$

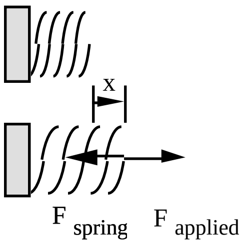
$\therefore$ $C = (9.8\ ms^{-2})(6.5\ 10^6\ m)^2 = 4.1\ 10^{14}\ m^3s^{-2}$

$$W = \int_{6500\ km}^{\infty} \frac{Cm}{r^2}\, dr$$

$$= -Cm\left(\frac{1}{\infty} - \frac{1}{6.5\ 10^6}\right)$$

$$= 6.3\ 10^{10}\ J \quad = 63\ GJ.$$

Work to deform spring



No applied force (x = 0)

Hooke's law: F = – kx

Work done **by** spring = $\int F_{spring}.dx$

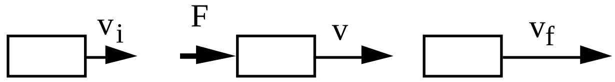
$$= \int -kx.dx = -\frac{1}{2}kx^2$$

Work done **on** spring = $\int F_{applied}.dx$

$$= \int kx.dx = +\frac{1}{2}kx^2$$ (= work stored in spring)

The work-energy theorem

(Total) force F acts on mass m in x direction.



Work done by F = $\int_i^f Fdx$ *(use F = ma)*

$$= \int_i^f m\frac{dv}{dt}dx = \int_i^f m\frac{dx}{dt}dv$$

$$= \int_i^f mv.dv = \left[\frac{1}{2}mv^2\right]_i^f$$

Work done by F = $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \equiv \Delta K$

Define **kinetic energy** $K \equiv \frac{1}{2}mv^2$

Increase in kinetic energy of body = work done by **total** force acting on it.

This is a theorem, ie a tautology
because it is only true by definition of KE and by Newton 2.
∴ restatement of Newton 2 in terms of energy. ***Not*** *a new law*

Power. is the rate of doing work

Average power $\bar{P} \equiv \frac{W}{\Delta t}$

Instantaneous power $P = \frac{dW}{dt}$

SI unit: 1 Joule per second $\equiv$ 1 Watt (1 W)

Example Jill (m = 60 kg) climbs the stairs in Matthews Bldg and rises 50 m in 1 minute. How much work does she do against gravity? What is her average output power? (neglect accelerations)

$$W = \int \underline{\mathbf{F}} \cdot \underline{\mathbf{ds}} = \int F_y \, dy$$

$$F_y \cong mg$$

$$W = mg \int dy = mg \, \Delta y$$

$$= 29 \text{ kJ}$$

(cf $K = \frac{1}{2}mv^2 \sim$ 20 - 40 J)

$$\bar{P} \equiv \frac{W}{\Delta t} = \frac{mg \, \Delta y}{\Delta t} = 490 \text{ W}$$

(humans can produce 100s of W, car engines several tens of kW)

(1 horsepower ≡ 550 ft.lb.s^{-1} = 0.76 kW)

Potential energy.

e.g. Compress **spring**, do W on it, but get no K. Yet can get energy out: spring can expand and give K to a mass. → Idea of stored energy.

e.g. **Gravity**: lift object (slowly), do work but get no K. Yet object can fall back down and get K.

Recall $W_{\text{against grav}} = mg \, \Delta y$ i.e. $W = W(y)$

But: Slide mass slowly along a surface. Do work against **friction**, but can't recover this energy mechanically. Not all forces "store" energy

Conservative and non-conservative forces

(same examples)

$$W_{\text{against grav}} = -\int_i^f F_g dr \cos\theta$$

$$= -\int_i^f F_g \, dz$$

$$= mg \int_i^f dz$$

$$= mg\,(z_f - z_i)$$

in uniform field

W is uniquely defined at all $\underline{\mathbf{r}}$, i.e. $W = W(\underline{\mathbf{r}})$

If $z_f - z_i$ are the same, W = 0.

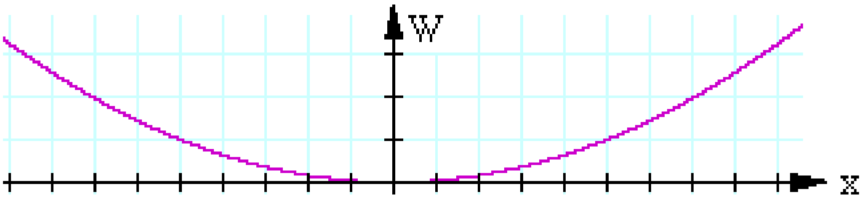
∴ Work done against gravity round a closed path = 0

Gravity is a **conservative force**

Compare spring

$$W_{\text{against spring}} = -\int_i^f F_{\text{spring}}.dx = -\int_i^f -kx.dx$$

$$= \frac{1}{2}k(x_f^2 - x_i^2)$$



W is uniquely defined at all x, i.e. W = W(x)

$x_f = x_i \Rightarrow W = 0.$

$\therefore$ Work done round a closed path = 0

Spring force is a **conservative force**

so it has stored or potential energy: symbol U.

with friction

$dW_{\text{against fric}} = - F_f \, ds \cos\theta$

but $\underline{\mathbf{F}}_f$ always has a component *opposite* $\underline{\mathbf{ds}}$

$\therefore$ dW always ≥ 0. *(we never get work back)*

$\therefore$ cannot be zero round closed path, $\therefore$ $W \neq W(\underline{\mathbf{r}})$

$\therefore$ friction is a **non-conservative force**

Note that direction of friction (dissipative force) is always against motion.

Potential energy

For a **conservative** force $\underline{\mathbf{F}}$ (i.e. one where work done against it, $W = W(\underline{\mathbf{r}})$) we can define potential energy U by $\Delta U = W_{\text{against}}$.
i.e.

$$\Delta U = -\int_i^f F \, dr \cos\theta$$

Same example: **spring**

$$\Delta U_{\text{spring}} = -\int_i^f F_{\text{spring}}.dx$$

$$= \frac{1}{2}k(x_f^2 - x_i^2)$$

Choice of zero for U is arbitrary.

Here U = 0 at x = 0 is obvious, so

$$U_{\text{spring}} = \frac{1}{2}kx^2$$

From energy to force:

$U = -\int F\, ds$ where ds is in the direction // F

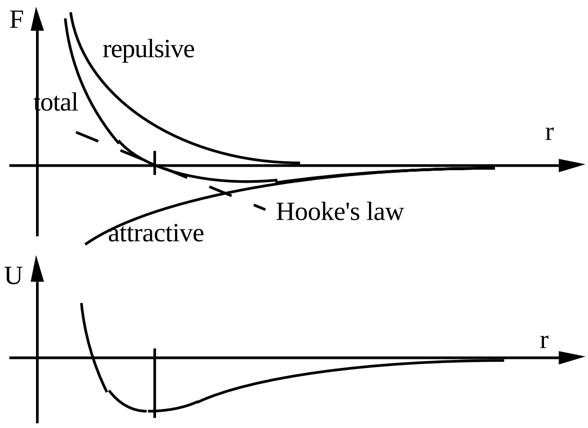
$$F = -\frac{dU}{ds}$$

in fact $F_x = -\frac{dU}{dx}, F_y = -\frac{dU}{dy}, F_z = -\frac{dU}{dz}$

Spring: $U_{spring} = \frac{1}{2}kx^2 \quad \therefore \quad F_{spring} = -kx$

Gravity: $U_g = mgz \quad \therefore \quad F_g = -\frac{dU}{dz} = -mg$

Energy of interaction:



Conservation of mechanical energy

Recall: Increase in K of body = work done by **total** force acting on it. *(restatement of Newton 2)*

But, if all forces are conservative, work done **by** these forces = $-\Delta U$ *(definition of U)*

$\therefore$ if only conservative forces act, $\Delta K = -\Delta U$

We define mechanical energy

$$E \equiv K + U$$

so, if only conservative forces act, $\Delta E = 0$.

we can make this stronger.

Work done by **non-conservative forces**

Define internal energy U_{int} where

ΔU_{int} = – Work done **by** n-c forces

(= + Work done **against** n-c forces)

Recall def^n of K: ΔK = work done by Σ force

$\therefore \quad \Delta K = -\Delta U - \Delta U_{int}$

$\therefore \quad \Delta K + \Delta U + \Delta U_{int} = 0$

If n-c forces do no work, then $\Delta U_{int} = 0$, so:

If non-conservative forces do no work,

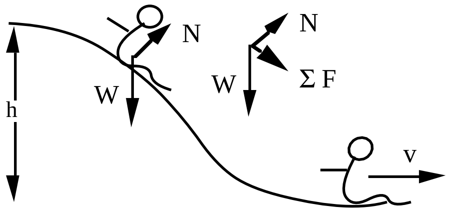
$$\Delta E \equiv \Delta K + \Delta U = 0$$

or: **mechanical energy** E **is conserved**

Equivalent to Newton 2, but useful for many mechanics problems where integration is difficult.

State the principle carefully! ***Never, ever write***: *"kinetic energy = potential energy"*

Classic problem. Child pushes off with v_i. How fast is the s/he going at the bottom of the slide? Neglect friction (*a non-conservative force*).



i) *By Newton 2 directly:*

$$v = \int_{top}^{bottom} a\,dt = \int_{top}^{bottom} \frac{F}{m} = \int_{top}^{bottom} g \cos\theta\,dt = ...$$

ii) *Using work energy theorem (Newton 2 indirectly):*

Non-conservative forces do no work, ∴ mechanical energy is conserved, i.e.

$\Delta E = \Delta K + \Delta U = 0$ *or* $E_f = E_i$

$K_f - K_i + U_f - U_i = 0$ $K_f + U_f = K_i + U_i$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 + mgy_f - mgy_i = 0$$

rearrange → $v_f = \sqrt{v_i^2 + 2g(y_i - y_f)}$

Conservation of energy

observation: for many forces, $W = W(\mathbf{r})$ ∴ useful to define $U = U(\mathbf{r})$.

observation: for all systems yet studied, U_{int} is a state function, i.e. $U_{int} = U_{int}$(measured variables)

Hence idea of internal energy. e.g.:

Friction, $(-U_{int})$ = heat produced when work is done against friction.

Air resistance $(-U_{int})$ is sound and heat.

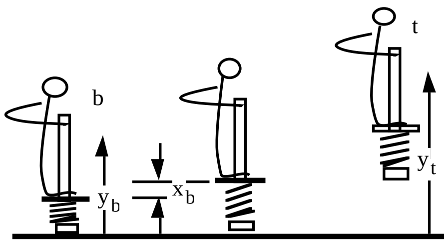
Combustion engines and animals: $+U_{int}$ comes from chemical energy

$$\Delta K + \Delta U + \Delta U_{int} = 0$$

is statement of Newton 2 plus definitions of K, U, U_{int}.

The statement that ΔU_{int} is a state function is the first law of thermodynamics. It is a law, ie falsifiable. More on this in Heat.

Example. Freda (m = 60 kg) rides pogo stick (m << 60 kg) with spring constant k = 100 kN.m^{-1}. Neglecting friction, how far does spring compress if jumps are 50 cm high?



Non-conservative forces do no work, ∴ mechanical energy is conserved, i.e.

$E_{bottom} = E_{top}$

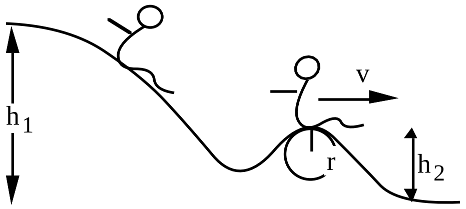
$K_b + U_b = K_t + U_t$ *($U = U_{grav} + U_{spring}$)*

$$\frac{1}{2}mv_{horiz}^2 + (mgy_b + \frac{1}{2}kx_b^2)$$

$$\cong \frac{1}{2}mv_{horiz}^2 + (mgy_t + \frac{1}{2}kx_t^2)$$

$$mg(y_t - y_b) \cong \frac{1}{2}kx_b^2$$

$$\therefore \quad x_b \cong \sqrt{\frac{2mg(y_t - y_b)}{k}} \cong 80 \text{ mm.}$$



Example. Slide starts at height h_1. Later there is a hump with height h_2 and (vertical) radius r. What is the minimum value of $h_2 - h_1$ if slider is to become airborne? Neglect friction, air resistance.

Over hump, $a_c = \frac{v^2}{r}$ (down). Airborne if $g < a_c$, i.e. if $v_2{}^2 > gr$.

No non-conservative forces act so

$$E_2 = E_1$$
$$U_2 + K_2 = U_1 + K_1$$
$$mgy_2 + \frac{1}{2}mv_2{}^2 = mgy_1 + \frac{1}{2}mv_1{}^2$$
$$\frac{1}{2}mv_2{}^2 = mg(y_1 - y_2)$$
$$(y_1 - y_2) = \frac{v_2{}^2}{2g} > \frac{gr}{2g} = \frac{r}{2}$$

Example Bicycle and rider (80 kg) travelling at 20 m.s^{-1} stop without skidding. $\mu_s = 1.1$. What is minimum stopping distance? How much work done by friction between tire and road? Between brake pad and rim? Wheel rim is ~300 g with specific heat ~ 1 kj.kg^{-1}, how hot does it get?

friction → decleration → stopping distance

$$|a| = \frac{F_f}{m} \leq \frac{\mu_s N}{m} = \mu_s g$$
$$|a| \leq \mu_s g$$

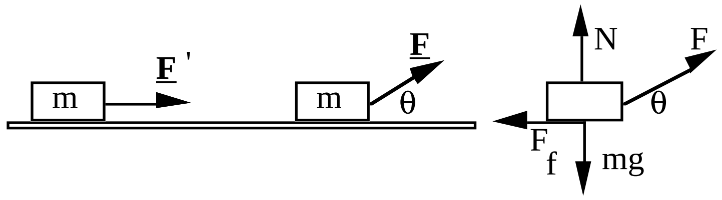
$$v_f{}^2 - v_i{}^2 = 2as \quad \rightarrow \quad s = \frac{v_f{}^2 - v_i{}^2}{2a}$$

$$s \geq \left|\frac{-v_i{}^2}{2\mu_s g}\right| = 19 \text{ m}$$

Work done by friction between tire and road?
No skidding, ∴ no relative motion, ∴ W = 0.
Between pad and rim? Here ∃ relative motion.
All K of bike & rider → heat in rim and pad

$$W = \Delta K = K_f - K_i = -\ 16 \text{ kJ}$$
$$Q = mC\Delta T \ \ \ldots\ldots \ \ \Delta T \sim 50\ °C$$

Example Which way is it easier to drag an object?



Suppose we move at steady speed, a = 0. Which requires less F? Which requires less work?

mechanical equilibrium $\rightarrow$ horizontal $F\cos\theta = F_f$

vertical $N + F\sin\theta = mg$

sliding $\rightarrow$ $F_f = \mu_k N$

$F\cos\theta = \mu_k N$

eliminate N $\rightarrow$ $F\cos\theta = \mu_k(mg - F\sin\theta)$

$$F = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta}$$

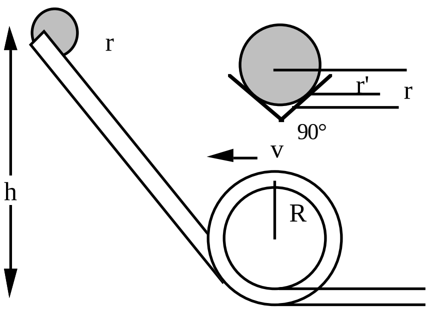
when $\theta = 0$, $F' = \mu_k mg$

$F < F'$ if $\cos\theta + \mu_k\sin\theta > 1$, i.e. if μ_k large & θ small

Work done = $Fs\cos\theta = F_f s$

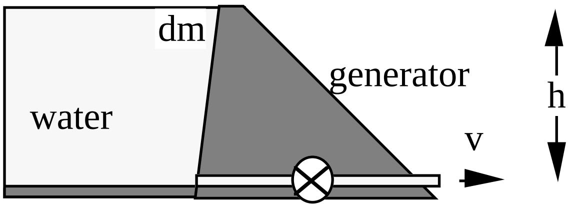
$= \mu_k Ns = \mu_k s(mg - F\sin\theta)$ *decreases* with θ

Question



How high should h be so that it can loop the loop?

Example. A hydroelectric dam is 100 m tall. Assuming that the turbines and generators are 100% efficient, and neglecting friction, calculate the flow of water required to produce 10 MW of power. The output pipes have a cross section of 5 m^2.



Nett effect: ~ stationary water lost from ***top*** of dam, water appears with speed v at bottom.

Let flow be $\frac{dm}{dt}$.

$$dW \equiv \text{work done by water}$$

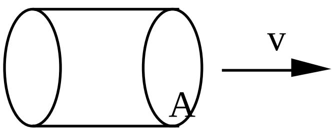
$$= - \text{Work done on water} = - \text{energy increase of water}$$

$$dW = - dE = - dK - dU$$

$$= - \left(\frac{1}{2} dmv^2 - 0\right) - \left(0 - dm.gh\right) = dm\left(gh - \frac{v^2}{2}\right)$$

$$P = \frac{dW}{dt} = \frac{dm}{dt}\left(gh - \frac{v^2}{2}\right)$$

Problem: v depends on $\frac{dm}{dt}$



v.dt

$$\frac{dV}{dt} = \frac{A.(v.dt)}{dt} = Av$$

Density: $\rho \equiv \frac{\text{mass}}{\text{volume}} = \frac{m}{V}$ so $m = \rho V$

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = \rho Av$$

$$P = \rho Av\left(gh - \frac{v^2}{2}\right)$$

$$v^3 - (2gh)v + \frac{2P}{\rho A} = 0$$ *can solve cubit, but messy*

Neglect $v^3 \rightarrow v = \frac{P}{gh\rho A} = 2$ m/s

and indeed we see that $v^3 <<$ *other terms*