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The nonlinear physics of musical instruments

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Abstract

Musical instruments are often thought of as linear harmonic systems, and a first-order description of their operation can indeed be given on this basis, once we recognise a few inharmonic exceptions such as drums and bells. A closer examination, however, shows that the reality is very different from this. Sustained-tone instruments, such as violins, flutes and trumpets, have resonators that are only approximately harmonic, and their operation and harmonic sound spectrum both rely upon the extreme nonlinearity of their driving mechanisms. Such instruments might be described as 'essentially nonlinear'. In impulsively excited instruments, such as pianos, guitars, gongs and cymbals, however, the nonlinearity is 'incidental', although it may produce striking aural results, including transitions to chaotic behaviour. This paper reviews the basic physics of a wide variety of musical instruments and investigates the role of nonlinearity in their operation.

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Contents

1. Introduction

Musical instruments have been of interest to scientists from the time of Pythagoras, 2500 years ago, and since then many famous physicists, among them Helmholtz, Rayleigh and Raman, have devoted at least some of their attention to them. For those interested in the early history of acoustics, there is an excellent book by Hunt (1978) and a collection by Lindsay (1972) of notable papers in the field. Musical instrument acoustics is well served by a number of general books, including those by Benade (1976), Sundberg (1991) and Taylor (1992), while a thorough mathematically based treatment with extensive references is given by Fletcher and Rossing (1998). Bowed-string instruments have been discussed in detail by Cremer (1984), and an extensive collection of reprints has been edited by Hutchins (1975-6) and by Hutchins and Benade (1997). A similar collection on the piano and wind instruments has been edited by Kent (1977), and one on bells by Rossing (1984). A classic book on woodwinds, recently reprinted, is that of Nederveen (1969). There is also a great wealth of illustrated historical literature on the development of particular musical instruments over the centuries.

Most elementary treatments of the acoustics of musical instruments rely upon a linear harmonic approximation. The term 'linear' implies that an increase in the input simply increases the output proportionally, and the effects of different inputs are simply additive. This certainly seems reasonable—a violin playing loudly sounds very much like a louder version of a violin playing softly! The term 'harmonic' implies that the sound can be described in terms of components with frequencies that are integral multiples of some fundamental frequency, and indeed this pattern of small-integer frequency ratios provides the basis of harmony and melody in all Western music, not just for numerological but also for sound psychophysical reasons (Helmholtz 1877 ch 10, Sethares 1998). Once again, there is a simple reason for this harmonic assumption. It is known that the mode frequencies of a stretched string, a cylindrical air column, and a conical horn are harmonically related, and all that appears to be necessary is to couple one of these passive resonators to some sort of controllable energy source in the form of a frictional bow, an air jet, or a vibrating reed, so as to maintain its oscillations.

Only in the case of percussively excited instruments such as bells and gongs does it seem necessary to recognize that the modes are not harmonically related. The sounds of such instruments do not fit easily into our Western musical tradition, though they are the foundation of music such as that of the Javanese Gamelan. It is only recently, with the aid of computers and electronics, that the possibilities of inharmonically based music are being explored in a systematic manner (Sethares 1998).

All this seems straightforward enough until the physics is examined in a little more detail. It is then found that the mode frequencies of a real string are not exactly harmonic, but relatively stretched because of stiffness (Morse 1948), and that the mode frequencies of even simple cylindrical pipes are very appreciably inharmonic because of variation of the end-correction with frequency (Levine and Schwinger 1948). Despite this, the sounds produced by mechanically bowed violins or by blown organ pipes are precisely harmonic, as can be seen by the fact that the waveform remains unchanged for hours, implying harmonicity to better than 1 part in 10^6. This is not so much important of itself as it is indicative of a deep underlying physical principle. Indeed, it is found that sustained-tone instruments can appropriately be described as 'essentially nonlinear', for it is nonlinearity that binds the sound together into harmonicity.

The situation with impulsively excited instruments is very different. The sound produced by a plucked or struck string is not exactly harmonic, and allowance for this must be made in the tuning of pianos, which have a scale that is stretched by nearly half a semitone across their compass (Schuck and Young 1943). Bells, gongs and cymbals, of course, have very inharmonic mode frequencies, and this provides their characteristic sounds. All this is very nearly linear, albeit inharmonic, behaviour, and these instruments are appropriately described as 'incidentally nonlinear'. This incidental nonlinearity can, however, have very striking effects when the level of impulsive excitation is large.

It is the purpose of this review to examine what has been discovered about musical instruments in recent years, using these observations as a background. The linear approximation to the physics of musical instruments still tells us a great deal, and we shall give it due weight, but much of the interesting physics and musical utility derives from nonlinearity. As in most studies of nonlinear phenomena, the field is advancing rapidly, and treatments in terms of Poincaré sections, Lyapunov exponents and correlation dimensions are beginning to appear (Müller and Lauterborn 1996, Wilson and Keefe 1998), but the approach adopted here will concentrate instead on the basic physics involved.

2. Sustained-tone instruments

It is helpful to consider the whole system that constitutes a sustained-tone musical instrument and its player, as shown in figure 1. The instrument itself generally has a primary harmonic resonator that is maintained in oscillation by a power source provided by the player, together with a secondary resonator, generally with some broad and inharmonic spectral properties, that acts as a radiator for the oscillations of the primary resonator.

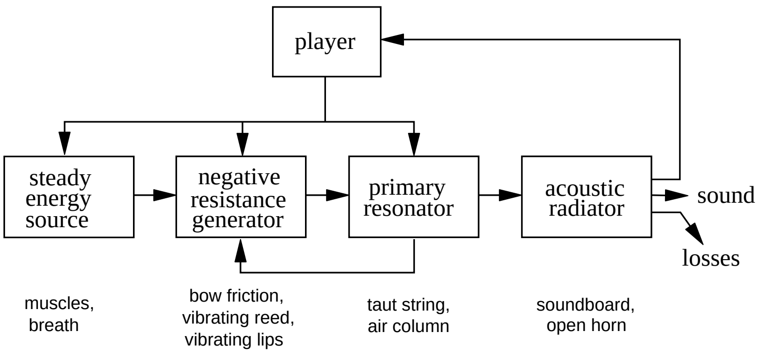

Figure 1. System diagram for a sustained-tone musical instrument. In most cases the generator is highly nonlinear and all the other elements are linear.

In the linear harmonic approximation, the generator is assumed simply to provide a negative resistance of limited magnitude that is sufficient to overcome the mechanical and acoustic losses in the primary resonator, but we can see that this provides us with no information about the spectral envelope, and thus the tone quality, of the sound produced. If, however, we make an assumption about this envelope, such as a uniform decline of 6 dB/octave in the case of a bowed string, then this spectrum will be modified by the vibrational and radiational properties of the secondary resonator and radiator—the body of a string instrument—and this will determine the overall tone quality. The difference between a good violin and a poor one arises primarily from the vibrational properties of the instrument body. We return to this in more detail in the next section. Wind instruments are rather different, in that the primary resonating body is the air column, which also radiates the sound. There is therefore one less element in the total system diagram.

In the more detailed nonlinear treatment, we recognize that the generator itself is usually highly nonlinear, and that the coupling between it and the primary resonator is so close that they cannot be considered separately. We also recognize that the primary resonator is usually appreciably, and often markedly, inharmonic in its modal properties. The feedback coupling between the resonator and the generator therefore assumes prime importance in determining the instrument behaviour.

In the following sections we shall discuss the various classes of musical instruments, initially in the linear harmonic approximation and then at the full nonlinear inharmonic level, so as to appreciate the vital role of nonlinearity in determining their acoustic behaviour. It is useful first, however, to examine the effects of inharmonicity and nonlinearity in quite a general fashion so as to see what is to be expected.

Before embarking upon this enterprise it is useful to consider one additional factor, and that is the overall efficiency of a musical instrument in converting muscular power into sound output. The acoustic output of a typical musical instrument ranges from a few tens of microwatts to several hundred milliwatts, corresponding to an on-axis sound pressure level of 60 to 110 dB at 1 metre, the upper range applying mainly to brass instruments. The dynamic range of most instruments is not much more than 20 dB, but the clarinet, trumpet and trombone have a range of rather more than 30 dB (Meyer 1978, Eargle 1990). Muscular power input, which can be measured by the product of bow friction and bow velocity for string instruments, or the product of blowing pressure and air flow in wind instruments, typically ranges from a few hundred milliwatts in strings and woodwinds up to as much as 10 watts in brass instruments. The overall efficiency of the instrument, from an acoustical point of view, ranges from about 0.1% to about 5%. Sound output is almost a minor by-product of the total instrument system!

3. Inharmonicity, nonlinearity and mode-locking

The primary resonator of any musical instrument, as noted above, can only be approximately harmonic in its mode frequencies, and many resonators, such as the tube with open finger holes found in woodwind instruments, have quite markedly inharmonic resonances. It is important to examine the coupling of this resonator to a negative-resistance generator that maintains it in oscillation, and to understand the effects of nonlinearity.

Figure 2(a) shows the input impedance (or perhaps input admittance, depending upon the type of generator, as is discussed later) for a typical slightly inharmonic resonator. Only four resonance peaks are shown, with peak frequencies ω_1, ω_2, ω_3 and ω_4, but it can be assumed that the curve extends to higher frequencies. The free oscillation of this resonator is a superposition of the normal modes $y_n(t)$, each of which obeys an equation of the form

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} + \alpha_n \frac{\mathrm{d}y_n}{\mathrm{d}t} + \omega_n^2 y_n = 0 \tag{3.1}$$

where ω_n is the natural frequency and α_n the damping coefficient of mode n. In the presence of a generator, the output of which depends upon the way it is excited by the normal vibrations of the resonator, the equation becomes

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} + \alpha_n \frac{\mathrm{d}y_n}{\mathrm{d}t} + \omega_n^2 y_n = g(y_1, y_2, y_3, \ldots) \tag{3.2}$$

where $g(y_1, y_2, \ldots)$ describes the driving force contributed by the generator. If we initially assume that the generator is linear, then we can write

$$g(y_1, y_2, y_3, \ldots) = c_1(y_1 + y_2 + y_3 + \ldots) + c_2(\dot{y}_1 + \dot{y}_2 + \dot{y}_3 \ldots) \tag{3.3}$$

where $\dot{y} \equiv \mathrm{d}y/\mathrm{d}t$ and c_1 and c_2 are constants, though perhaps involving a phase shift in complex cases.

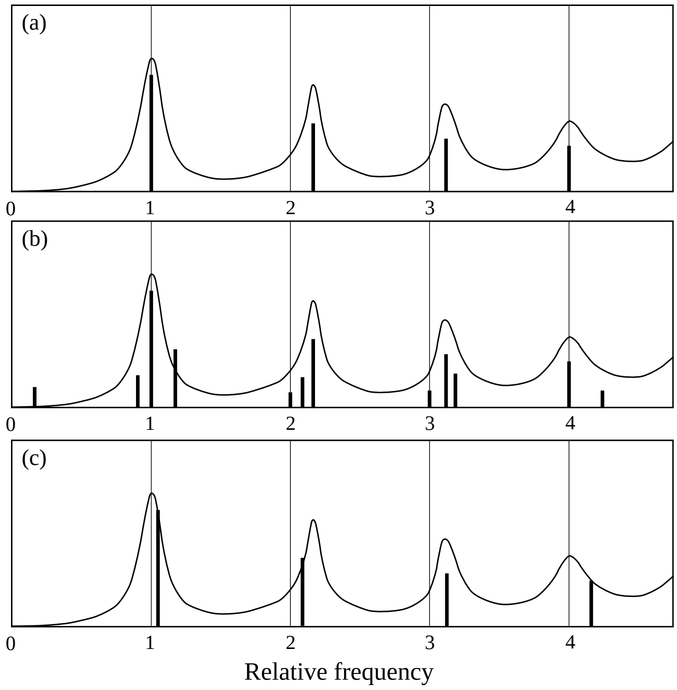

Figure 2. (a) The input impedance (or admittance, depending upon the system) of a slightly inharmonic resonator, together with the power spectrum produced when it is excited by a generator presenting a simple linear negative resistance. (b) The multiphonic power spectrum of the same resonator when driven by a generator with a slightly nonlinear characteristic. (c) The mode-locked response of the same resonator when driven by a highly nonlinear generator. Note the small frequency shift of the fundamental and its harmonics.

If we write the mode displacements in the form

$$y_n(t) = a_n \sin(\omega_n t + \phi_n) \tag{3.4}$$

then mode n is driven primarily by the generator term $c_1 y_n + c_2 \dot{y}_n$, because it matches it in frequency. If the part $c_2 \dot{y}_n$ of g that is in-phase with $\mathrm{d}y_n/\mathrm{d}t$ is greater than the damping term $\alpha_n \dot{y}_n$, then the mode amplitude a_n will grow, until it is limited by some assumed increase of the damping coefficient α_n or decrease in the generator force $c_2 \dot{y}_n$—a necessary nonlinearity! The part $c_1 y_n$ of g that is in-phase with y_n will cause some small shift in the oscillation frequency away from ω_n. If the generator provides a pure negative resistance, however, then $c_1 = 0$ and this shift will be zero. The final power spectrum of the driven resonator will then consist simply of lines at the resonance peaks, as in figure 2(a).

In the real situation, however, the generator is always nonlinear, for physical reasons that will be discussed later in connection with particular musical instruments. This means that the generator response function (3.3) must be supplemented by terms of the form $c_{nm} y_n y_m$ and even higher orders such as $y_n^p y_m^q \ldots$ and similar terms involving the $\dot{y}_n$. These contribute driving terms with frequencies $\omega_n \pm \omega_m$, or generically $p\omega_n \pm q\omega_m \pm \ldots$, where $p, q, \ldots$ are integers. Each primary mode excitation is thus surrounded by a host of 'combination tones' as shown in figure 2(b). Such an acoustic output is generally referred to as a 'multiphonic'. Multiphonics find use in some contemporary music, and are also produced accidentally by beginning players!

Examination of the power spectrum in figure 2(b) shows that it can be regarded as a set of 'carrier' oscillations at frequencies ω_n, with each carrier being accompanied by 'side-bands'. As in radio communications, we can represent this situation in terms of carrier waves that are modulated in amplitude and in frequency or phase. Analytically this means that the mode excitations given by equation (3.4) should be generalised by taking both amplitude and phase to be functions of time, so that

$$y_n(t) = a_n(t)\sin[\omega_n t + \phi_n(t)]. \tag{3.5}$$

If the inharmonicity is not too great, then the side-bands will be not too far away from ω_n, and $a_n(t)$ and $\phi_n(t)$ will vary with time at a rate that is much less than ω_n. This 'method of slowly varying parameters' for the treatment of nonlinear problems was introduced by Bogoliubov and Mitropolsky (1961), and an account of the approach is also given by Morse and Ingard (1968). It has been applied to the musical instrument situation by Fletcher (1978).

To proceed, we substitute (3.5) back into (3.2) and simplify by retaining only slowly varying terms. This results in the relations

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \left\langle \frac{g(y)}{\omega_n}\cos(\omega_n t + \phi_n)\right\rangle - \frac{\alpha_n a_n}{2} \tag{3.6}$$

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}t} = -\left\langle \frac{g(y)}{a_n\omega_n}\sin(\omega_n + \phi_n)\right\rangle \tag{3.7}$$

where $y = \sum y_n(t)$ and the brackets $\langle \ldots \rangle$ imply that only terms varying slowly relative to ω_n are retained, for example by averaging over one period of the oscillation. These expressions are also true for the linear case, as indeed they must be for consistency.

The implications of equations (3.6) and (3.7) are clear. The amplitude of each of the modes will change continuously, as also will the effective frequency ω_n', which is given by

$$\omega_n' = \omega_n + \mathrm{d}\phi_n/\mathrm{d}t. \tag{3.8}$$

The rate and extent of change in both amplitude and frequency will increase as the nonlinearity of the function g, and thus the amplitude of the nonlinearly generated terms at frequencies near ω_n, increases. Provided the nonlinearity is great enough, however, the system is likely to pass through a state in which the frequencies of two large-amplitude modes n and m are momentarily in simple small-integer relationship. When this happens, the phases of the two modes can lock and their amplitudes adjust so that $\mathrm{d}a_n/\mathrm{d}t = \mathrm{d}a_m/\mathrm{d}t = 0$. Other weaker modes are then either recruited to this phase-locked harmonic regime or else eliminated, so that the whole oscillation becomes strictly harmonic.

It is not easy to define the sufficient conditions for this to happen, though necessary conditions involve a proportionality between the degree of nonlinearity in the generator and the degree of inharmonicity in the resonator. Numerical integration of the equations (3.6) and (3.7) for all the modes involved, however, shows that the behaviour depends upon the initial conditions, as is indeed found in practice when playing musical instruments. A computed example, in this case for the onset transient of a mildly inharmonic organ flue pipe, is shown in figure 3 (Fletcher 1976b). The wide oscillations in mode amplitude and frequency are clearly shown, together with the ultimate stabilization into the phase-locked harmonic regime. From this discussion it is clear that adequate nonlinearity is essential for the production of harmonic tones from sustained-tone instruments.

This treatment of nonlinearity and transients has been couched in terms of normal modes and their evolution. We might term this a frequency-domain treatment. McIntyre *et al* (1983) (see also Woodhouse 1995) have presented a different approach that is formulated completely in the time domain. This is particularly attractive when it is desired to determine the output

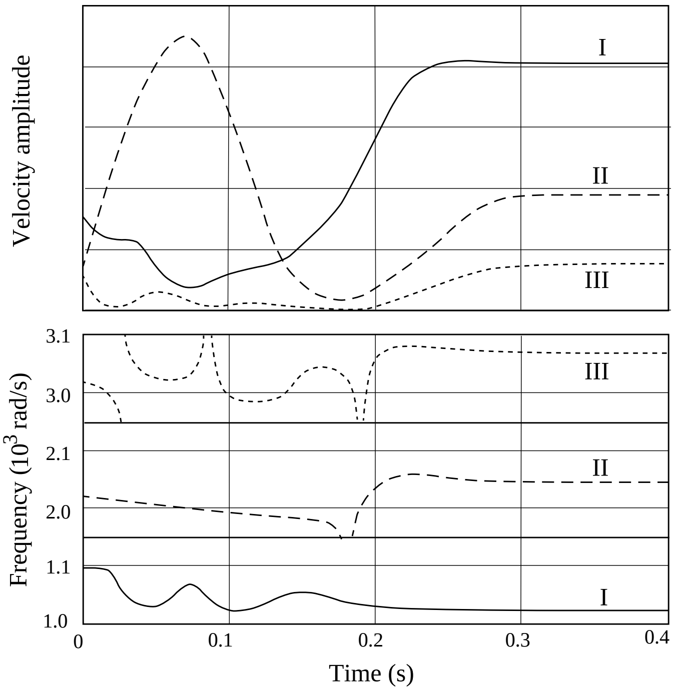

Figure 3. Calculated initial transient for a flue organ pipe following a plosive onset of blowing pressure. The upper panel shows the amplitudes and the lower panel the frequencies for the first three modes, labelled I, II and III respectively. Note the broken frequency scale (Fletcher 1976b).

waveform from a particular instrument, because it is easily adapted to computational use. It is also excellent for calculating initial transients and other time-varying phenomena. Its disadvantage, from out present point of view, is that its physical interpretation is less clear, because the modes of the primary vibrator enter only indirectly.

In this approach, we define the physics of the nonlinear generating element by giving its time-varying output $g(y, t)$ as a function of an arbitrary time-varying input $y(t)$. This is essentially our nonlinear generator function $g(y)$ of equation (3.3), except that the oscillator quantity $y(t)$ is not now broken up into a sum of contributions from normal modes. The oscillator is similarly not described in terms of its normal modes but rather by specifying its impulse response $G(t - t')$ at the point of connection to the generator. This impulse response is formally the Fourier transform of the input impedance of the resonator. The time behaviour of the system can then be expressed as

$$y(t) = \int_0^t G(t - t')g(y, t')\mathrm{d}t' \tag{3.9}$$

and this can be integrated numerically once we have specified the initial conditions and, if necessary, the time-dependence of the parameters in the generator function g. There are complications with this approach because the impulse response function $G(t - t')$ generally has a very long extension in time, corresponding to the decay time of oscillations in the resonator. This can be avoided by reformulating the problem in terms of a reflection function at the input rather than an impulse response function, as discussed by Ayers (1996) for the case of wind instruments. This frequency-domain approach will not be followed here because we wish to place the emphasis on normal modes and their interactions.

We now go on to discuss both the linear properties of the resonators and the nonlinear properties of the generators that constitute real musical instruments.

4. Bowed-string instruments

Of all the instruments of Western music, the family of bowed strings is perhaps the most important and most studied. It consists of just four instruments—violin, viola, cello and double bass—and their form has been well established for something like 300 years. They evolved from the earlier family of viols but, except for the double bass, which retains the flat back and often the five strings of the bass viol, the evolution produced a very different instrument with arched top and back plates, no frets on the fingerboard, and a newly designed bow. The instruments produced by the Italian masters of the 17th century in Cremona—Amati, Stradivari and others—are taken to define the style and quality that modern instruments aim to reproduce.

Research, as documented in the reprint collections edited by Hutchins (1975-6) and by Hutchins and Benade (1997), concentrates on the violin, but similar principles apply to other instruments. The present review can discuss only a small fraction of the work that has been done.

4.1. Linear harmonic theory

The linear harmonic model has guided a good deal of contemporary violin research, the object of which has been to discover the design secrets of the old Italian masters so that modern violins with similar sound can be built. An acoustic study is necessary because each piece of wood is different, so that a simple copying of geometrical dimensions is not an adequate approach. Rather, it has proved necessary to study the vibrational modes of the violin body, both during construction and after completion, and to match these to the modes of the instrument being copied. Traditional violin making does indeed incorporate this practice in a simplified manner, since the maker holds the top-plate near one edge and taps it to identify the pitch of the ring-tone produced, different tap positions exciting different modes.

In a research environment, the violin top-plate, which is its most critical element, is typically examined by some form of laser vibrometry as it is excited sonically from below (Hutchins *et al* 1971, Hutchins 1981). Such a study yields not only the frequencies of the plate modes but also their mode shape, which is important because the body will be excited in a particular place defined by the feet of the bridge over which the strings pass. Violin makers in a more practical environment may sprinkle fine ‘glitter’ flakes or sand over the vibrating plate to achieve a similar result by collecting at the nodal lines. The thickness of the arched plate can then be adjusted by removing wood from its lower surface so that both the mode frequencies and mode shapes approximate those of the instrument being copied, or some other tonal ideal. Similar methods are applied to the back plate.

When the violin is assembled, the mode frequencies change because of the clamping and inter-plate coupling conditions applied by the ribs around the plate edge and by the sound-post underneath the bridge. There are also air modes to be considered, in particular the Helmholtz cavity mode associated with air flow through the *f*–shaped tone holes, but also modes with higher wave numbers inside the body cavity. These air modes are coupled to the plate modes because plate modes cause volume or shape changes in the cavity.

The basic physics of these mode interactions was worked out by Schelleng (1963) using electric network analogs that are now common in analysing acoustic systems. Similar techniques can be applied to higher modes. The result is a complex frequency envelope that describes the behaviour of the violin body. There are many ways in which this could be defined and measured, but the most practically meaningful is the transfer function between a force applied to the bridge and the acoustic pressure measured in the far field. A reverberant environment is required for this latter measurement because the various body modes all have different directional radiation characteristics. In practice a simpler transfer function is usually adequate, particularly if the properties of instruments are being compared. The easiest such measurement to interpret is the mechanical admittance (the ratio of velocity to force) at the bridge of the instrument where the string rests. Jansson (1997) has measured and compared such admittance curves for a representative group of high quality violins, while Saldner *et al* (1997) have shown how the body modes couple to the sound radiation field.

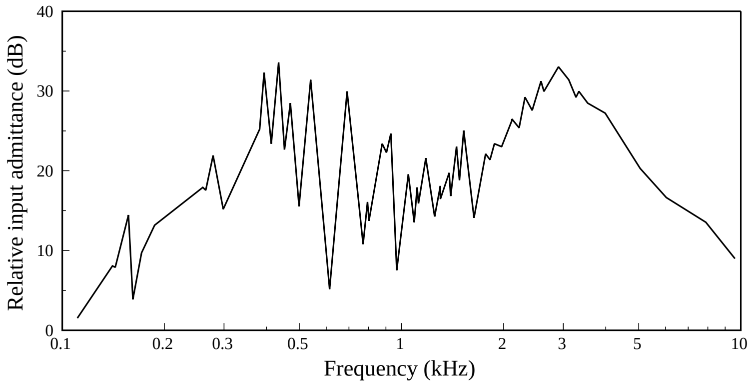

Figure 4. Resonance curve for a typical violin body, here defined as the mechanical input admittance at the bridge, or the ratio of velocity to force at this point. Note the body resonances at low frequencies and the general shape of the curve at higher frequencies.

An example of such an admittance curve for a good violin is shown in figure 4. The lowest peak is associated with the air mode coupled in-phase with the lowest body mode, which produces a large net change in cavity volume. Higher resonances are associated with an out-of-phase coupling of these two modes and with direct coupling to higher body modes. The essence of effective plate tuning is that the prominent lower resonances of this curve should be well distributed in frequency, bearing in mind however that the string produces a rich harmonic excitation and that the human auditory system can generate a false fundamental when presented with strong upper partials. The variation of tone quality with pitch in this lower range is, indeed, part of the attraction of violin tone, provided that particular notes do not sound unduly weak or unduly dominant.

The spectral envelope of the curve generally shows a dip between 1 and 2 kHz in the case of a good violin, a broad peak around 3 kHz, and then a smooth decline at higher frequencies. This envelope, which is characteristic of old Italian violins, effectively defines ‘good’ classical violin quality, though violinists from environments such a folk music may prefer a brighter sound associated with greater extension of the envelope to higher frequencies. This feature is, at least in part, a function of the internal damping of the wood from which the violin is made.

When the input impedance curves of other bowed-string instruments are compared with that of the violin, or indeed when their characteristic tone qualities are examined critically, it is clear that they are not simply larger and lower-pitched versions of the violin, but have their own musical signatures. Composers have indeed exploited these differences in writing for the viola, the cello and the double bass, but another school of thought, initiated by Carleen Hutchins (1967, 1992) has sought to design a family of instruments that match the acoustical properties of the violin, though at a series of higher and lower pitches. Several sets of eight instruments have been designed and built according to this principle: one violin a fifth and one an octave higher in pitch than the standard violin and five larger instruments separated in pitch by fifths and fourths, though with the lowest two having their strings tuned in fourths, like the double bass, rather than in fifths like the higher instruments. Acoustically these instruments are very successful, as can be judged by listening to recordings (St Petersburg Octet 1998). Their acceptance has been slow, however, because professional players are understandably wedded to the playing response of their own traditional instruments and to their characteristic tonal qualities.

4.2. Nonlinear bowed-string generators

While the vibrational behaviour of the body of a bowed-string instrument is very closely linear, the frictional mechanism that drives the bowed string itself is highly nonlinear. The general form of the motion was first studied experimentally by Helmholtz (1877) and a descriptive theory was developed by Raman (1918). Detailed modern treatments have been given by McIntyre and Woodhouse (1979, 1995) and by Cremer (1984). In this review we can hope only to outline the more interesting features.

First consider not a string but a simple oscillator consisting of a mass and spring, with the mass excited by a moving frictional belt, as shown in figure 5(a). If the moving belt is brought into gentle contact with the stationary mass, then the friction between them will cause a displacement of the mass and initiate an oscillation at the resonance frequency. If the speed of the belt is high and the force of friction is independent of the relative velocity of the belt and mass, then the amount of energy supplied to the oscillator in its forward motion will be exactly equal to the frictional loss in the reverse motion, but the oscillation will be damped by internal losses in the spring and decay to zero. If, however, the coefficient of friction increases with decreasing relative velocity, as shown in figure 5(b), then there will be a net supply of energy to the oscillator and the amplitude of the oscillation will grow, as shown in figure 5(c). This growth will continue until the maximum forward velocity of the mass equals the belt velocity, after which no further growth is possible because the sign of the frictional force abruptly reverses. As the energy of the vibrating mass increases, it will be constrained to stick to the belt (zero relative velocity) for a longer time and then to flick rapidly back to the opposite extreme of its motion under the influence of only the smaller sliding friction. The evolution of the velocity waveform is thus as shown in figure 5(c) and that of the displacement waveform in figure 5(d). It is clear that the characteristic squared-off stick-slip waveform depends upon the nonlinearity of the frictional force, and in particular upon the gross nonlinearity and sign change near the condition of zero relative velocity. We note in passing that, had the mass initially been in contact with a stationary belt that then began to move, the initial transient would have been different.

The motion of a bowed string is much more complex because of its linear extension, but resembles that discussed above in many of its features. In the ideal motion, the string shape consists of two straight-line sections joined at a kink, which moves with steady velocity around a double-parabolic path as shown in figure 6(a). This path, surprisingly, is independent of the point of application of the bow, though the direction of motion of the kink depends upon the direction of the bow motion. On this model, the motion of the string under the bow can be seen to consist of a period of constant-velocity slow motion in the direction of the bow (the sticking period) followed by a shorter slipping period in which the velocity is constant, higher, and in the opposite direction, as shown in figure 6(b). The ratio of the durations of these two periods is equal to the ratio of the two lengths into which the bow position (or, more generally,

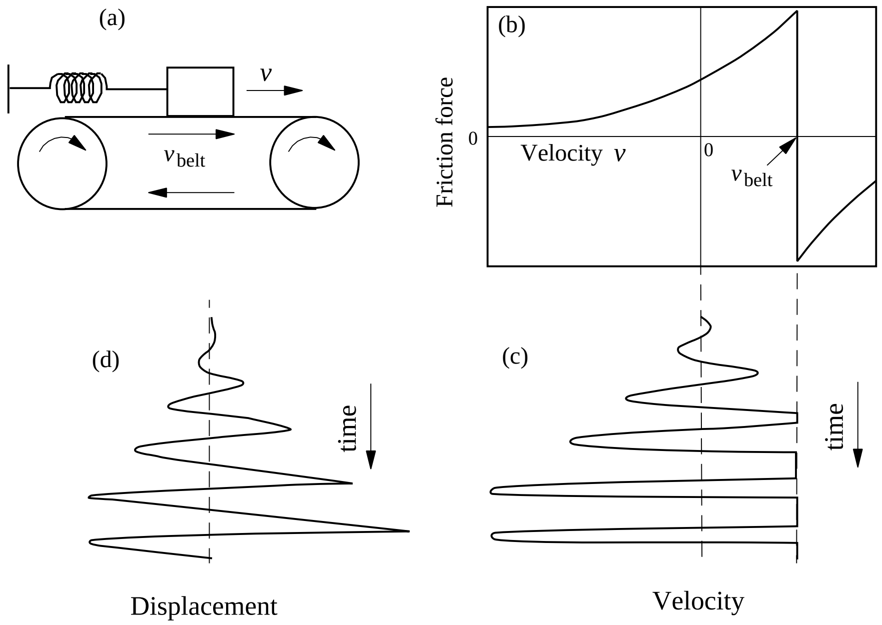

Figure 5. (a) A simplified mass-and-spring analog for bowed string motion. (b) The nonlinear behaviour of frictional force as a function of relative velocity. (c) The associated evolution of the velocity waveform. (d) The evolution of the displacement waveform.

the observation point) divides the string.

It must be admitted that this idealised model of bowed-string motion omits many features of importance (Woodhouse 1995). The stiffness of the string, for example, means that the kink is not ideally sharp; the non-zero diameter of the string means that torsional motion is also important; and the finite width of the bow complicates the capture and release transition. In real instruments therefore, particularly in the hands of unskilled players, the motion does not follow exactly the Helmholtz ideal. The slip-phase is not strictly of constant velocity, and the release event may not be ideally simple, so that in effect there may be two or more Helmholtz kinks following each other around the parabolic path. Some of these nonlinear features have been investigated by Müller and Lauterborn (1996) using a digitally simulated bowing force. Despite these complications, the simplified model does give a good understanding of the basic mechanism and of typical bowing technique.

The important thing from an acoustical point of view is the transverse force applied to the bridge by the vibrating string. This has the form of a sawtooth wave, as shown in figure 6(d), the bowing position now having no influence. Fourier analysis of this force waveform shows a spectrum containing all harmonics of the fundamental, with the amplitude of the nth harmonic varying as $1/n$, corresponding to a spectral envelope declining at 6 dB/octave. The regularity of this spectrum explains why a simple source plus linear resonator model works so well for bowed-string instruments.

Despite the apparent lack of influence of the bow on the driving force produced by the string, it does have an important role. In the first place, the amplitude of the string deflection is directly proportional to the bow speed and, for a constant bow speed, increases as the bowing

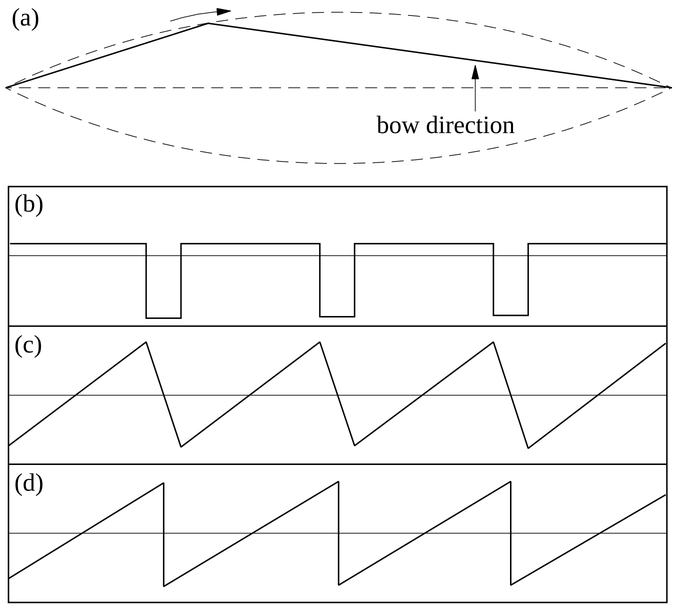

Figure 6. (a) Instantaneous configuration of a bowed string executing an ideal Helmholtz motion. The kink follows the parabolic dotted envelope in the direction shown for the bow direction indicated at the right-hand end of the string. (b) Velocity and (c) displacement waveforms at a position 0.2 of the length from one end of the string. (d) Transverse force waveform at the supporting bridge.

position approaches the end of the string. In compensation, there are both upper and lower limits to the bowing force required for any particular bow velocity if the standard Helmholtz stick-slip motion is to be achieved, and these limits both rise as the bowing position approaches the end of the string (Schelleng 1973, Askenfelt 1989).

In the case of a string to which a moving bow is applied, there is an initial oscillating transient with only mild nonlinearity before the highly nonlinear stick-slip regime sets in. During this initial transient, the modes of the string are not locked together, but the extreme nonlinearity of the fully developed motion rapidly achieves this mode locking. If, on the other hand, the bow is placed upon the string and then set into motion, the stick-slip mechanism is active from the beginning, though not initially in an exactly periodic manner.

The transient and steady motion of a bowed string is best calculated using the time-domain method of McIntyre *et al* (1983) discussed briefly in section 3. An excellent recent treatment has been given by Schumacher and Woodhouse (1995), using the massive parallel computing power of a Connection Machine to investigate a wide region of parameter space. As well as describing initial transients and the steady state, this method is well suited to the examination of peculiar oscillation regimes such as wolf-notes, in which a strong body resonance is coupled to the string vibration and causes it to be split into two, and situations in which nominally improper bowing pressures or bow speeds are used, giving multiple slip events in each period.

5. Wind instruments

Wind instruments of various types have a very long history. Many have evolved very significantly over the centuries, but some remain recognizably similar to their ancestors from a thousand years ago. An excellent summary of their more recent history has been given by

Carse (1939).

As noted in the introduction, the air column in a cylindrical pipe is only approximately harmonic in its resonances, because the end-correction at the open end reduces steadily with increasing frequency (Levine and Schwinger 1948). The same is true of an open conical pipe, complete to its apex. A realizable instrument, such as an organ pipe, has additional complications because its mouth is relatively constricted, giving a much increased end-correction and consequent greater progressive sharpness to the resonances. A conical pipe blown at the narrow end similarly must suffer truncation to allow connection to the sound generator, and this gives further inharmonicity (Ayers *et al* 1985). When woodwind instruments are considered, then the situation is made vastly more complicated by the presence of open or closed finger holes in the pipe wall—the first few resonances may be roughly harmonically related for notes in the lowest register, but the resonances become very complex and inharmonic for fingerings in the higher registers (Backus 1974). Indeed, the higher registers the instrument generally operates on a resonance that is well above the lowest pipe resonance. It is therefore at first sight quite surprising that the sounds they produce have strictly harmonic overtones (except for special 'multiphonic' effects), and we must look to nonlinearity to explain this.

In the case of woodwind instruments, different notes are produced by the simple expedient of opening finger holes in the side of the instrument tube. Matters are complicated by the fact that there are 12 semitone steps in an octave, while players have only 10 fingers, but this problem was initially overcome by various compromise fingerings, and in the past 200 years or so by the addition of padded keys to cover the holes and a complex mechanism to interlink them.

Brass instruments are somewhat simpler in their input impedance patterns, because they do not have finger holes, and the whole shape of the bore, and particularly of the flaring horn, is designed so as to make the resonances as nearly harmonic as possible, with the exception of the lowest resonance, which is generally ignored.

These facts then beg the question of how the player manages to select a particular resonance upon which to base the sound of the note. Once this has been solved, then, provided the exciting generator is sufficiently nonlinear, we can look to the general theory of mode locking discussed in section 3 to ensure that the instrument produces a harmonic sound. Clearly we need to focus much of our attention upon the generator mechanism.

There are three classes of wind instruments that require consideration: the standard reed woodwinds, oboe, clarinet, bassoon and saxophone, in which the exciting mechanism is a single or double cane reed; the brass instruments, trumpet, trombone and the like, in which the exciting mechanism is the player's vibrating lips; and the flute-family instruments in which excitation is produced by an air jet striking an edge in the mouthpiece of the instrument. We consider these in turn and identify both the nature of the generator mechanism and also the reason for its nonlinearity (Fletcher 1990). In all cases, however, the details of the aerodynamics are much more complex than appears at first sight (Hirschberg 1995). It is fortunate that a reasonably simple approach in which these subtleties are largely ignored actually gives a good account of the observed behaviour.

6. Woodwind reed generators

Most studies of reed woodwinds have been carried out on the clarinet, since its reed and bore geometries are simpler than those of the other instruments. Among the studies that should be mentioned are those of Backus (1963), Schumacher (1981) and Kergomard (1995).

The reed-generator of a clarinet has the general form shown in figure 7(a). It consists of a flat tapered reed held against an aperture in the slightly curved face of the mouthpiece in such a way that the static opening x_0 is about 1 mm. The blowing pressure p_0 inside the player's mouth forces air into the mouthpiece at a speed v determined by the Bernoulli equation

$$p_0 - p_1 = \tfrac{1}{2}\rho v^2 \tag{6.1}$$

where p_1 is the pressure inside the instrument mouthpiece, and ρ is the normal density of air. At the same time, the pressure difference $p_0 - p_1$ tends to close the aperture between the reed and the mouthpiece so that its area becomes

$$S = [x_0 - \beta(p_0 - p_1)]W \tag{6.2}$$

where β is the elastic compliance of the reed and W is its width. Putting these together, the volume flow U through the reed is

$$U = vS = \left[\frac{2(p_0 - p_1)}{\rho}\right]^{1/2}[x_0 - \beta(p_0 - p_1)]W. \tag{6.3}$$

This relation is plotted as a full line in figure 7(c).

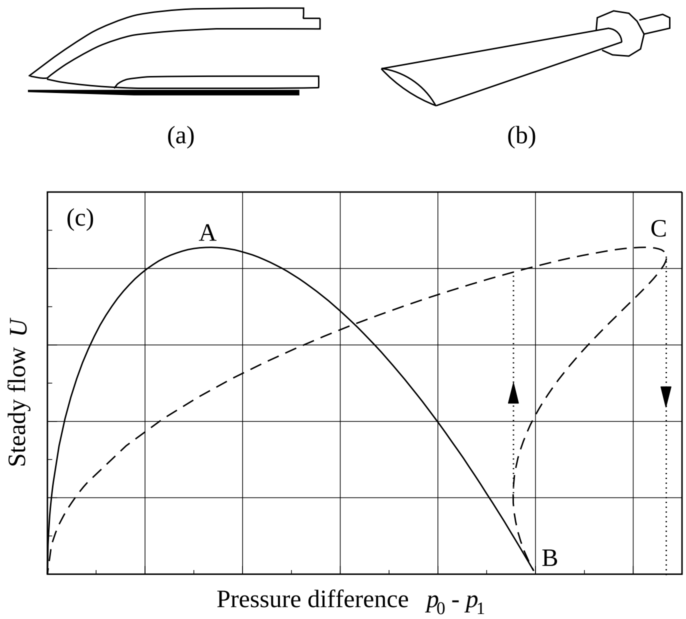

Figure 7. (a) The single reed of a clarinet or saxophone. (b) The double reed of an oboe or a bassoon. (c) The quasi-static flow characteristic of a single reed as in a clarinet (full curve), for which the series flow resistance $R \approx 0$, and of a double reed as in an oboe (broken curve).

It is helpful now to define the acoustic admittance presented to the instrument by the reed generator, which is given by the equation

$$Y = -\mathrm{d}U/\mathrm{d}p_1 \tag{6.4}$$

the negative sign arising since U has been taken to be the flow into, rather than out of, the instrument. In general there will be a phase factor involved, and Y will be a complex quantity, but in the quasi-static model considered here we assume that the resonance frequency of the reed is sufficiently high compared with the sounding frequency that there is no phase shift and Y is simply a conductance. Since Y is just the slope of the curve in figure 7(c), it is clear that this conductance is negative in the region AB of the curve, and thus above the pressure p_A

which is just one-third of the pressure required to completely close the reed. The reed can thus act as an acoustic generator provided the blowing pressure p_0 is greater than p_A.

This all works out properly for a clarinet, or for a saxophone which uses a similar reed and mouthpiece. The resonance frequency of the reed is typically around 2000 Hz and the playing frequency less than 1000 Hz so that the quasi-static approximation is justified. The pressure required to close the reed is usually about 6 kPa (60 cm water gauge), and a player typically uses blowing pressures in the range 2 to 5 kPa, nearly independently of the pitch of the note being played. To control the loudness of the sound produced, the player controls lip tension, and through it the static reed opening x_0. For a high lip pressure, x_0 is reduced, and with it the whole scale of the curve in figure 7(c), leading to a softer sound.

Because the characteristic in figure 7(c) is not straight, the generator output is a nonlinear function of the mouthpiece pressure p_1 and produces multiple sum and difference frequencies in the flow U as discussed in section 3. In the case of a clarinet, the input acoustic impedance Z_{in} of the air column is high near the frequencies of odd harmonics of the fundamental and low near the frequencies of even harmonics, at least in the low register. The pressure $p_1 = Z_{in}U$ fed back to control the reed motion thus contains primarily odd harmonics, and the oscillation locks into a regime in which these are emphasized. In the upper register, however, the mode frequencies are sufficiently inharmonic that preference for odd harmonics is lost. A saxophone, on the other hand, has a bore in the shape of a slightly truncated cone, the resonances of which include both odd and even multiples of the fundamental frequency, so that the sound also has this structure.

For single-reed instruments, particularly the clarinet when played softly, the nonlinearity is not extreme and multiphonic sounds that are not mode-locked can be produced when unusual fingerings are used. These multiphonics consist of multiple sum and difference frequencies, as usual, and are employed in some contemporary music (Bartolozzi 1981).

The way in which the player manages to select a particular resonance upon which to base the sounding frequency also deserves attention. The quasi-static approach outlined above is adequate for low notes, but becomes questionable at higher frequencies. It is possible, however, to treat the reed as an independent vibrator with its own resonance properties and, on this basis, to derive the behaviour of the reed generator (Fletcher 1979). The results are shown in figure 8. Phase shift in the reed response to the exciting pressure causes the admittance to follow a circular path in the complex plane. If the blowing pressure is below the quasi-static generating threshold, then the conductance is negative only over a region just below the reed resonance. For higher blowing pressures the conductance is negative at all frequencies below the reed resonance, but there is a preferred oscillation region just below reed resonance. The player can vary the reed resonance to some extent by lip position and, by aligning the preferred oscillation region with the desired instrument resonance or perhaps with a low multiple of this frequency, can select this as the operating mode. Clearly such subtle control requires effort and experience to achieve.

There are many subtle complexities to the physics of these single-reed instruments. The sizes of the finger holes (covered by large pads in the case of the saxophone) influence the general spectral envelope of the radiated sound (Benade 1960, Benade and Kouzoupis 1988) and the saxophone mouthpiece must be designed so that its internal volume compensates for the volume removed by truncating the instrument cone (Benade and Lutgen 1988).

In the case of double reeds such as those of the oboe or bassoon, as shown in figure 7(b), an extra complication enters (Hirschberg 1995, Wijnands and Hirschberg 1995). Because of the geometry of the reed, there is a narrow flow channel downstream of the reed tip opening, and this contributes a flow resistance. Because the flow velocity is rather fast, we might expect the pressure drop across this resistance to have the form RU^2, rather than varying linearly with U.

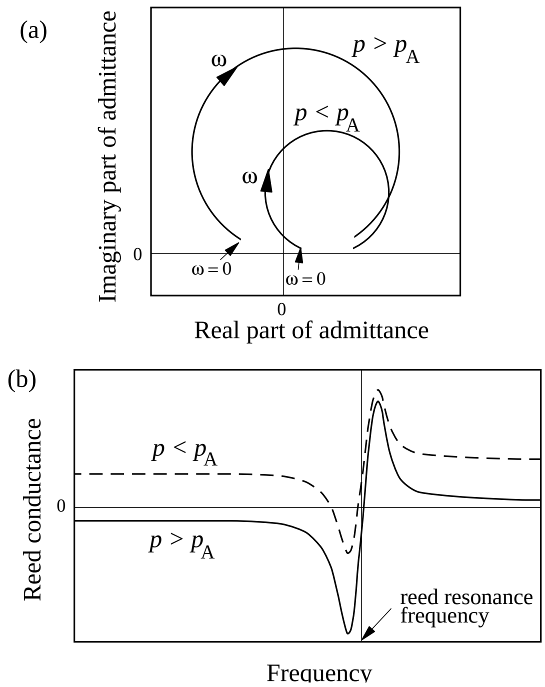

Figure 8. (a) Complex admittance of a woodwind-type single reed generator as a function of frequency for blowing pressures below and above the low-frequency operating threshold p_A. (b) Conductance of a single-reed generator as a function of frequency for blowing pressures below and above the low-frequency operating threshold p_A.

The effect of this resistance can be calculated by replacing p_1 by $p_1 - RU^2$ in equation (6.3). The result is the highly nonlinear flow characteristic shown as a broken curve in figure 7(c). The threshold pressure for generator activity is now p_C, which is typically 4–7 kPa for the oboe and 2–6 kPa for the bassoon, depending upon pitch and loudness, and the nonlinearity is so extreme that the reed follows a path like that shown dotted, closing fully in each cycle. This accounts in part for the characteristic sound of double-reed instruments.

The spectral envelope of a reed-woodwind playing a note of fundamental frequency ω_1 can be described by an expression of the form

$$p(n\omega_1) \sim \frac{[U(\omega_1)/U_0]^n Z_{\text{in}}(n\omega_1)}{1 + (n\omega_1/\omega_c)^m} \tag{6.5}$$

where $U(\omega_1)$ measures the volume flow through the reed at the fundamental frequency ω_1, and thus essentially the loudness of the sound, $Z_{\text{in}}(\omega)$ is the input impedance of the instrument at frequency ω, and ω_c is the open-tone-hole cutoff frequency above which waves will propagate without reflection along the bore (Benade 1960, Benade and Kouzoupis 1988). The envelope of this expression, for the case of a clarinet, is shown in figure 9. Typically $\omega_c \approx 1500$ Hz for treble instruments such as the clarinet and oboe. U_0 is a limiting flow, determined by a curve such as that in figure 7(c), that $U(\omega_1)$ does not exceed. For a non-beating reed, $m = 6$, which is reflected in the spectral envelope as a fall of 18 dB/octave above ω_c, while for a beating reed $m = 4$, giving a fall of 12 dB/octave. These figures reflect the nature of the flow discontinuity when the reed is nearly closed. For a clarinet, Z_{in} is large for odd harmonics, declining slowly with increasing frequency, and small for even harmonics, rising slowly with frequency, and this is reflected in the limiting spectrum for loud playing. For an oboe, with its conical bore, $Z_{\rm in}$ is large for all harmonics of the fundamental and may even rise slightly with frequency. Because the oboe reed beats, the decline above $\omega_{\rm c}$ is only 12 dB/octave, giving a brighter sound. For soft playing on the clarinet, the factor $[U(\omega_1)/U_0]^n$ comes into effect and the spectrum declines quite rapidly, even below $\omega_{\rm c}$, but since an oboe reed always beats this factor is essentially unity in its case at all playing levels.

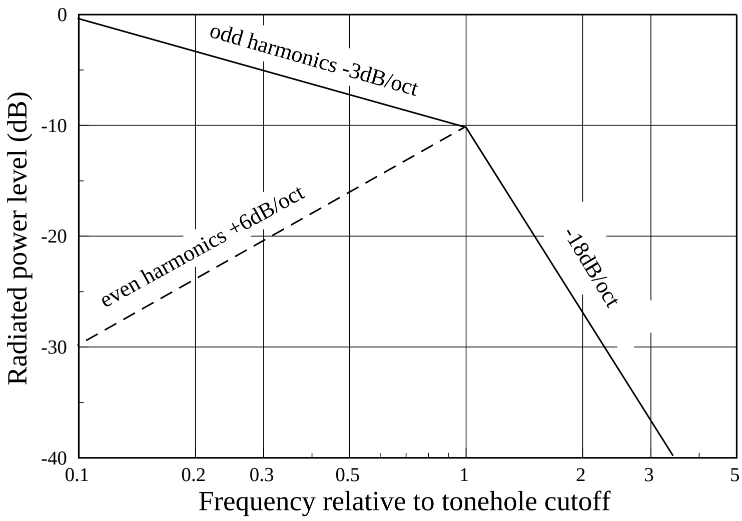

Figure 9. The limiting spectral envelope for a clarinet playing loudly. Separate lines are shown for the odd and even harmonics. For softer playing, the spectrum lies below these lines. If the reed beats against the mouthpiece, the high-frequency fall is only 12 dB/octave.

Organ-pipe reeds have some elements of similarity with the reeds of clarinets, in that they are single brass tongues that are blown closed against a slotted brass tube known as a shallot. Their behaviour is, however, quite different from woodwind reeds in that it is the reed, rather than the pipe resonator, that controls the pitch of the note sounded. The pipe, which may either be a conical horn of half wavelength or a cylindrical tube of one-quarter wavelength, is first tuned to the pitch to be sounded. The reed pitch is then gradually raised by shortening its vibrating length by means of a clamping wire. The pipe begins to sound weakly at a frequency well below its nominal pitch, and the sound increases to reach full strength when reed and pipe are essentially in tune with each other. If the reed pitch is raised further, then the pipe ceases to sound. While there have been detailed expositions of reed-pipe geometry and voicing practice published (e.g. Hesse and Furtwängler 1998), there has been little formal study. The behaviour can, however, be readily understood on the basis of our discussion above.

If the reed is blown at a pressure that is just insufficient to bring its quasi-static conductance into the negative (generating) region, then, from the upper curve in figure 8(b), there will still be negative acoustic conductance over a narrow region just below the frequency of the reed resonance. Because the reed is made of metal and not damped by the presence of a player's lips, this resonance will be quite sharp and the negative conductance large. The impedance of the pipe will assist oscillation of this type of reed provided it is inertive (Fletcher 1993), which requires that the sounding frequency be somewhat below that of a pipe resonance. Just above the pipe resonance, the pipe impedance is compliant and opposes the reed oscillation. For correct adjustment of the pipe therefore, the sounding frequency should be just below the reed resonance and just below the pipe resonance. Higher harmonics of the reed fundamental are generated in the flow and amplified by the pipe resonator, as with woodwind instruments, but have little effect on the reed motion because it is so close to resonance.

7. Brass instruments

Brass instrument horns have an approximately harmonic mode structure which is maintained over most of their compass, since there are no finger holes to interfere. The lower modes are widely spaced and are used to produce the familiar bugle-calls. Higher modes, particularly those above the eighth, produce an approximation to a diatonic major scale, although adjustment of pitch is needed either in the bore design or in playing technique to make this approximation adequately good in 'natural' instruments without valves or slides. The slide of the sackbutt, the ancestor of the modern trombone, allows precise adjustment of mode frequencies at the expense of some lack of agility, while most brass instruments use a set of valves to add extra lengths to the cylindrical part of the bore. This approach necessarily leads to compromise, since the practice on instruments such as trumpets is to use only three valves, and when they are combined the combination is an additive rather than a multiplicative one—if $(1+x)$ is the length ratio for a semitone step, then addition of $2x$ gives a length rather less than $(1+x)^2$.

The design of an instrument horn to produce a complete series of nearly harmonic mode frequencies clearly relies upon the skill of its maker, backed by centuries of tradition. Since the player's lips constitute a pressure-controlled valve, they must operate at a pressure maximum in the horn, and its mouth is, of course, open. Early instruments such as the cornett, serpent and ophicleide, had a conical horn with finger-holes, like a woodwind instrument, and naturally possessed a complete harmonic series of resonances, disturbed only by truncation at the mouthpiece. Modern brass instruments, however, all have a more-or-less cylindrical section of bore, terminated by a section that may be initially conical but then flares more rapidly than this at the mouth. The cylindrical part by itself would lead to an odd-harmonic resonance series, and proper design of the expanding part of the bore is necessary. By carefully proportioning the profile, the horn resonances can be brought into a good approximation to a harmonic sequence 2,3,4,5,... except for the fundamental, which typically has a relative frequency around 0.7 in this sequence and is not used in playing. The physics underlying horn design has been discussed in detail by Benade and Jansson (1974) and by Jansson and Benade (1974), relying upon the Webster equation for wave propagation in a duct of varying cross-section $S(x)$, namely

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial p}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{7.1}$$

where c is the velocity of sound in air. Interest centres upon the behaviour of waves in the flaring horn, which reflects low-frequency waves at an earlier point of its bore than high-frequency waves. Indeed the analysis shows that, if we define a 'horn function' $F = (1/r)\mathrm{d}^2r/\mathrm{d}x^2$, where r is the horn radius at position x, measured along a spherical wavefront, then waves with wavenumber $k = \omega/c$ are reflected at the point where $F = k^2$. Since, in this spherical-wave approximation, F has a maximum value F_{max}, waves with $\omega > cF_{\mathrm{max}}^{1/2}$ are freely transmitted.

In essentially all instruments there is a cup-like mouthpiece with a rather narrow back-bore section connecting it to the main instrument tube. Not only does this cup provide support for the player's lips, but its Helmholtz-like cavity resonance, loaded by the inertance of the back-bore and the characteristic impedance of the main horn, provides an overall envelope to the impedance maxima presented to the player's lips (Benade 1973) as shown in figure 10.

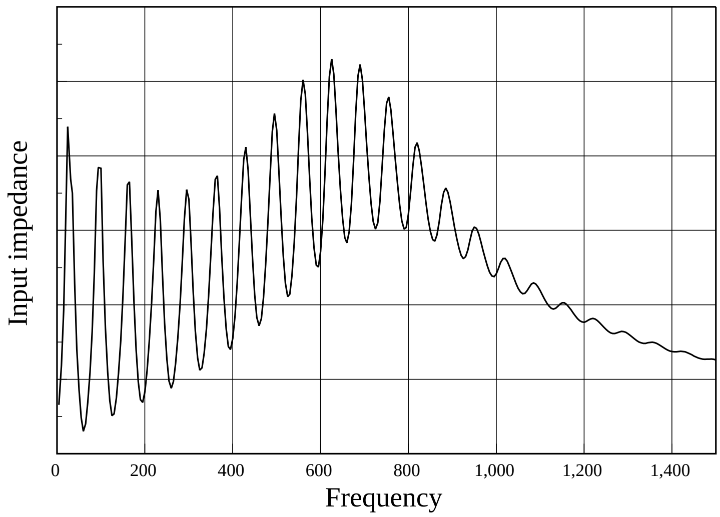

Figure 10. Input impedance of a typical brass instrument. The overall envelope is determined by the cavity resonance of the mouthcup, loaded by the inertance of the back-bore and the resistive characteristic impedance of the instrument horn.

Variations in mouthpiece design have a significant effect on tone quality (Wright and Campbell 1998).

The lip-valve mechanism that generates the sound has much in common with the reed valves of woodwinds, but also some significant and important differences. The first of these is that, while the blowing pressure in the player's mouth tends to close a woodwind reed valve, a situation we may denote by (−,), it tends to blow open the valve formed by a brass-player's lips, denoted by (+,). Because the lips are formed from soft tissue, their motion may be complicated, but in a simple approximation they may be considered to move either outwards towards the instrument, or else up-and-down in their own plane. These two motions have different characteristics when the effect of the pressure in the instrument mouthpiece is considered. If the lips move outwards under blowing pressure, then the mouthpiece pressure will tend to close them, (, −), while if the lips move in their own plane, the mouthpiece pressure will tend to open them, (, +) If the woodwind reed is described as an 'inward-swinging door' and denoted by (−, +), then these two alternatives may be described as an 'outward-swinging door' (+, −) and a 'sliding door' (+, +) respectively. Alternative terms are 'blown closed', 'blown open' and 'blown sideways'.

Defining the acoustic admittance of the lip-valve generator as $-\mathrm{d}U/\mathrm{d}p_1$, where p_1 is the pressure in the mouthpiece cup, we find that the sliding-door valve (+, +) behaves rather like the inward-swinging woodwind valve generator and has a preferred oscillation regime just below its resonance frequency, as in figure 8. The outward-swinging door valve (+, −), on the other hand, behaves in quite a different way, as shown in figure 11. This type of valve can act as a generator only in a narrow frequency range just above the lip resonance. The two configurations are compared in figure 11(b). Because the blowing pressure tends to blow open either type of lip valve, the phase of the mouthpiece pressure variation is such as to reduce the damping of the lip vibration (Fletcher 1993), so that the generator peak is quite sharp and oscillation takes place close to its mechanical resonance frequency. This also allows autonomous vibration of the lips under the influence of mouth pressure, even in the absence of

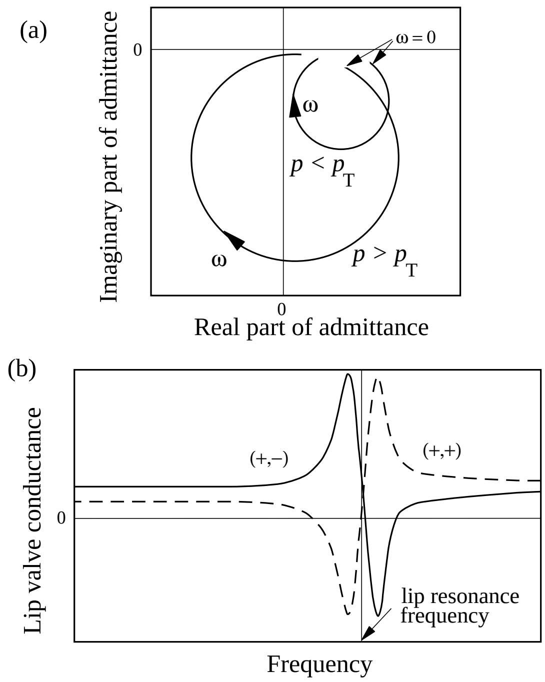

Figure 11. (a) Complex admittance of a lip-valve generator of the 'outward-swinging door' or (+, −) type, as a function of frequency, for a blowing pressures below and above the threshold p_T. (b) Acoustic conductance of a lip-valve generator, as a function of frequency, for a blowing pressure above the oscillation threshold. The full curve is for an 'outward-swinging door' (+, −) configuration and the broken curve for a 'sliding door' (+, +) configuration.

an instrument horn. This is important, for it is exactly what the lips must do for the first few cycles of any high note, before a reflection returns from the bell after a time about equal to the period of the horn fundamental.

Careful measurements (Yoshikawa 1995, Copley and Strong 1996) show that the lips of brass players tend to favour the outward-swinging door (+, −) behaviour at low frequencies and the sliding-door behaviour (+, +) at high frequencies, but this may depend upon the player and the instrument. Adachi and Sato (1996) have recently produced a more realistic lip model in which the lips are allowed two degrees of freedom, corresponding to the outward-swinging and sliding-door possibilities. This model combines the two motions and gives a generator peak that is closer to the lip resonance frequency than either of the elementary models. Perhaps even more realistic would be a model incorporating wavelike motion on the lips (Ayers 1998).

The pressure and flow waveforms in the mouthpiece of a brass instrument have been studied experimentally by Elliott and Bowsher (1982) and show very marked nonlinearity, despite the fact that the player's lips are observed to move nearly sinusoidally. The reasons for this are apparent when the air flow is examined in detail. Suppose that the lip opening is of constant width W and sinusoidally oscillating height x. Then the volume flow into the instrument mouthpiece is

$$U = (2/\rho)^{1/2} W x (p_0 - p_1)^{1/2} \tag{7.2}$$

where p_0 is the blowing pressure, p_1 the back-pressure in the instrument mouthpiece cup, and ρ the density of air. If we make the simplifying assumption that the instrument sounds exactly at one of its resonance frequencies, so that its input impedance is a simple resistance R, and further assume that R is the same at all harmonics of the sounding frequency, then the back pressure $p_1 = RU$. Substituting this back into equation (7.2) leads to a quadratic in U, which can be solved to give

$$U = \left(\frac{RW^2x^2}{\rho}\right)\left[\left(1 + \frac{2\rho p_0}{R^2W^2x^2}\right)^{1/2} - 1\right]. \tag{7.3}$$

This expression is clearly a quite nonlinear function of the lip opening x, and indeed if we assume that $x = a_0 + a\sin\omega t$, with a approaching a_0 in magnitude, then we can approximate (7.3) by

$$U \approx \frac{p_0}{R} - \frac{p_0^2}{R^3(a_0 + a\sin\omega t)^2}. \tag{7.4}$$

The flow predicted by the complete equation (7.3) is shown for increasing values of dynamic level and blowing pressure in figure 12, with the assumption that the lip vibration amplitude is proportional to blowing pressure and that the lips nearly close. This waveform is very much like the mouthpiece pressure waveform measured by Elliott and Bowsher (1982), and shows a marked increase in harmonic content with increasing lip vibration amplitude, and thus with sound level. The details can, of course, be modified, for example by assuming that the width W of the lip opening varies periodically in time with the variation in x, rather than remaining constant. This refinement was used by Fletcher and Tarnopolsky (1999), who assumed that $W = \gamma x^{1/2}$ with γ constant, and found that this gave a good fit to their experimental data.

It should be emphasized that this is by no means a complete model of sound production. For such a model, we must also write down an equation for the resonant driving of the lip vibration by the fluctuating mouth and mouthpiece pressures (Fletcher 1979, 1993, Adachi and Sato 1996). Such modelling procedures are now well developed.

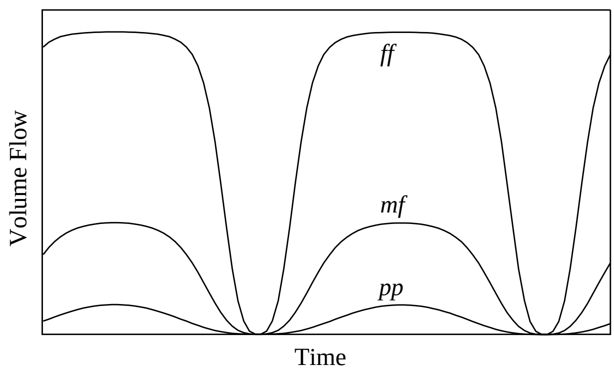

Figure 12. Volume flow through the lips of a trumpet player for increasing blowing pressure and lip vibration amplitude, as described by equation (7.3).

The radiated sound is, however, very much richer in high harmonics than would be expected from the waveform of figure 12, even when allowance is made for the fact that radiation resistance rises as the square of the frequency up to the radiation cutoff at about 1000 Hz. This is particularly noticeable in loud playing, when the spectrum extends to very high frequencies, as shown in figure 13. In an earlier study, Beauchamp (1980) examined the relation between radiated sound pressure and mouthpiece pressure and found that the radiated pressure above cutoff was highly variable, depending upon playing technique, and the radiated power at frequencies above this cutoff often appeared to exceed the input power. The explanation for this can be found in nonlinear propagation behaviour in the instrument bore itself, as was explained by Hirschberg *et al* (1996).

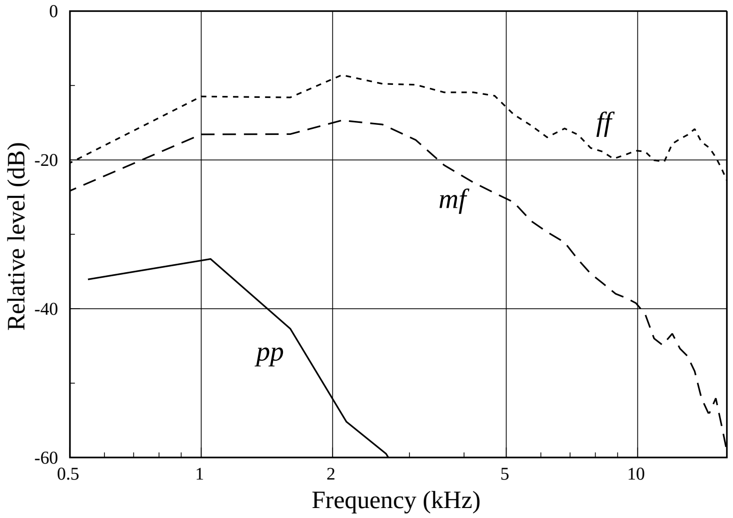

Figure 13. Envelope of the radiated sound spectrum of a trumpet playing the note C_5 (523 Hz) at three different loudness levels. The blowing pressures are *pp*: 3.3 kPa, *mf*: 6.3 kPa and *ff*: 13 kPa (Fletcher and Tarnopolsky 1999).

The blowing pressure used by expert trumpet players in loud playing is extremely high, sometimes reaching as much as 25 kPa, which is even greater than systolic blood pressure (Fletcher and Tarnopolsky 1999). In brass instruments such as trumpets and trombones, but not in the gentler-toned cornet and tuba, the bore is cylindrical for more than half the length of the instrument. In loud playing, the sound pressure in the instrument mouthpiece cup can reach levels as high as 175 dB, which is about 20 kPa, and may even somewhat exceed the blowing pressure. The sound pressure in the instrument bore will be less, but probably exceeds 2 kPa. At these high sound pressures, nonlinear terms such as the convective term $v\partial v/\partial x$, where v is the acoustic particle velocity, that are normally neglected in the wave equation become significant. The result is that the leading edge of the waveform of the acoustic flow, shown in figure 12, is progressively sharpened as it traverses this cylindrical tube, with a consequent transfer of energy from low to high harmonics. At extremes, a shock wave may even form (Hirschberg *et al* 1996). This acoustic nonlinearity in the air column allows trumpets and trombones to become the most assertive voices in the orchestra when the need arises. Although similar wavefront sharpening may occur in other brass instruments, it is generally less extreme because of their gradually expanding bore.

8. Flutes and organ flue pipes

There is another class of wind instruments in which the sound is generated by the action of an air jet meeting a sharp edge, without the need for any sort of mechanical valve. In this group we find the flue organ pipes, recorders, flutes, and whistles of many types. In organ pipes, the resonator is usually a simple open cylinder, with a mouth cut into one side just above the pipe foot and a slit at the lower edge of the mouth to produce a planar air jet that flows across the mouth to meet the upper lip. Some organ pipes, however, have stopped cylindrical pipes, so as to support mainly odd harmonics of the fundamental, while some have complex shapes with a chimney in an end stopper, designed to emphasise particular harmonics in the tone. Panpipes are simple collections of small stopped pipes that are excited by blowing over the open ends, and again these are simple because there is just one note to be produced from each pipe.

Flute-type instruments are more complex, because they must have finger holes to produce different pitches. The principle here is just the same as for the other woodwinds, and there has been an evolution from simple small finger holes to larger holes covered by padded keys and with linking mechanisms to produce semitones. Details are complicated, but the principles are relatively simple (Boehm 1871).

To understand sound production in these instruments, it is necessary first to consider the behaviour of air jets. This topic, investigation of which began with Rayleigh (1894), is still far from completely understood. In most of the instruments we consider, the air stream issues from an aperture that is slit-like, even if it is the player's lips, so that it is reasonable as a first approximation to consider the behaviour of an infinite plane jet of air moving through an infinite air space. Rayleigh showed that such a situation is unstable, in that an infinitesimal disturbance of the motion of the jet will grow to become large. Two different classes of disturbance can be identified: in one, which could be called 'sinuous', the jet is displaced sideways and then follows a curved path, while in the second, termed 'varicose', the jet is thickened in some places and thinned in others. The obvious way in which to treat these instabilities is to consider the behaviour of wave-like disturbances of defined wavelength and the appropriate type of jet displacement.

Rayleigh (1894) showed that, for a jet with top-hat velocity profile and thickness $2b$, moving with velocity V through still air, the phase velocity u of the disturbance behaves like

$$u_{\text{sinuous}} = \frac{V}{1 + \coth kb} \qquad u_{\text{varicose}} = \frac{V}{1 + \tanh kb} \tag{8.1}$$

where $k = \omega/u$ is the wave number of the disturbance. Thus $u_{\text{sinuous}} \to 0$ and $u_{\text{varicose}} \to V$ as $kb \to 0$, while both velocities approach $V/2$ for $kb > 1$. The amplitude of the disturbance also grows exponentially as $\exp \mu x$, where the growth constant μ is almost equal to k and thus increases without limit at high frequencies.

A real jet has a bell-shaped $\text{sech}^2(y/b)$ rather than a top-hat velocity profile, and this leads to significant differences in behaviour that are more accurately described (Nolle 1998) by the spatial analysis of Mattingly and Criminale (1971) rather than the temporal analysis of Drazin and Howard (1962) as used in earlier publications. The calculated behaviour of the wave velocity u and the growth factor μ for a sinuous jet disturbance is shown in figure 14. The difference from the behaviour of a top-hat profile jet is due to the lack of a sharp edge to the flow. The most important thing to note is that wave growth on the jet occurs only for $kb < 1.33$, and the exponential growth parameter μ has a maximum near $kb = 0.3$. Disturbances with wavelengths less than about one full jet width are therefore damped out. The wave velocity rises steadily with increasing frequency, but is essentially confined to the range $0.4V$–$0.6V$.

In practice, while jets do behave this way for small disturbances, once the amplitude of the disturbance becomes greater than the jet width there is a tendency for the jet to shed vortices in a regular manner, generating a 'vortex street'. Fortunately we can achieve a reasonable understanding of the role of jets in musical instruments using the small-amplitude theory (Coltman 1968, Elder 1973, Fletcher 1976c), for it appears that the main role of vortices, when they occur, is in dissipating energy and mixing the jet flow with the main flow in the instrument (Verge *et al* 1994, Fabre *et al* 1996), both of which processes can be treated in a simplified way.

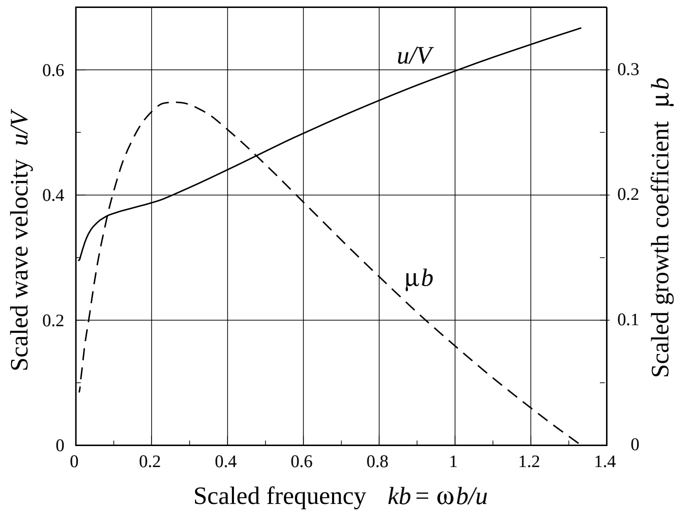

Figure 14. Growth parameter μ and wave speed u for sinuous disturbances on a jet with velocity profile $V\mathrm{sech}^2(y/b)$, as calculated by Nolle (1998). The scaled frequency parameter is 2π times the conventional Strouhal number.

Varicose jet oscillations are not used in musical instruments, though they are an efficient means of sound production, as demonstrated by various aperture whistles (Chanaud 1970) and indeed by human whistling, in which a jet, produced towards the back of the mouth and intercepted by the lip opening, develops varicose instabilities that interact with the Helmholtz resonance of the vented mouth cavity. The sounding frequency is determined by the mouth resonance, and the blowing pressure and mouth geometry must be adjusted so that there is about one half-wavelength of the disturbance on the jet between the initial aperture forming it and the intercepting aperture at the lips.

In the case of instruments employing sinuous jets, the jet emerges from a slit-like flue aperture and is intercepted by an edge or ‘lip’, lying parallel to the jet width, and coupled to a resonator. Only in the case of the relatively quiet sound made by an edge-tone mechanism, in which the feedback from the edge to the jet at the aperture is direct (Powell 1961), is a resonator not involved. In normal air-jet instruments, periodic deflection causes the jet to blow alternately into and outside the resonator, reinforcing its oscillations, while the acoustic flow from the resonator in turn influences the flow of the jet as it emerges from the slit. Because there is an appreciable time delay in travel of the deflection wave from the flue slit to the lip, there will be a limited range of jet velocities for which the total phase delay leads to resonator reinforcement, and therefore a limited range of operating blowing pressures.

The way in which the acoustic flow from the resonator through the pipe mouth and past the flue slit influences the jet is not immediately clear, and several models have been proposed. That which seems most reasonable, and most closely supported by experimental data (Nolle 1998), is the ‘negative displacement’ model of Fletcher (1976a). In this model the air flow past the flue displaces the body of the jet, but is unable to displace the jet where it emerges from the flue. This is the same as imposing a negative displacement, equal in amplitude to the acoustic displacement in the resonator flow, on the jet at the flue so that this part of the jet remains stationary. This negative displacement then propagates along the jet towards the lip and is amplified by the jet instability mechanism. If the acoustic flow velocity past the flue is

$v \exp j\omega t$, then this leads to a jet deflection of the form

$$y(x,t) = -\frac{jv}{\omega}\left\{\exp j\omega t - \cosh \mu x \left[j\omega\left(t - \frac{x}{u}\right)\right]\right\} \tag{8.2}$$

where u is the phase velocity on the jet and μ the jet amplification factor, as before. The amplification has been given a hyperbolic rather than an exponential form so that the jet angle is zero at the flue, but this is not crucial. This expression, perhaps modified to allow for some spreading and slowing of the jet, allows us to calculate the phase relation between the resonator flow and the jet deflection at the edge. The transit time along the jet typically introduces a delay of around a millisecond in addition to the phase delay associated with the interaction mechanism.

When the jet flows in an alternating fashion into and out of the pipe at the lip, it contributes to the acoustic flow in two ways (Coltman 1968, Elder 1973, Fletcher 1976a). Firstly, because the jet is rapidly slowed by turbulent interactions—and this is where the vorticity is involved—it transfers oscillating momentum to the flow in the pipe. Secondly, because there is an end-correction at the resonator mouth, there is also an acoustic pressure, and the volume flow of the jet must do work against this, depending again upon relative phase. If the component of the jet flow into the pipe at frequency ω is U_j, the end-correction at the mouth ΔL and the pipe cross-sectional area S_p, then the pipe flow U_p induced by the jet can be shown (Fletcher 1976a) to be

$$U_\mathrm{p} = \frac{(\rho V + j\rho\omega\Delta L)U_\mathrm{j}}{S_\mathrm{p} Z_\mathrm{s}} \tag{8.3}$$

where Z_s is the series acoustic impedance of the pipe and the end-correction at its mouth. The first term in the numerator describes the momentum drive and the second term the injected volume flow drive. Clearly the pipe oscillates most readily when it is in resonance, so that the magnitude of Z_s is a minimum. The end correction ΔL influences both the phase and magnitude of the response, through its influence on the relative proportions of the two types of drive.

This whole development can be captured graphically by plotting the complex admittance Y_j of the jet-lip combination, as viewed from the pipe resonator (Coltman 1968, Thwaites and Fletcher 1983). Such a plot is shown in figure 15. For any note there is a limited range of blowing pressure for which the phase relations in the jet mechanism will give a negative real part to Y_j and thus a positive feedback. Generally speaking, a flute player plays a high note by pushing the lips forward so as to reduce the jet length, and simultaneously increasing the blowing pressure and hence the jet velocity. The consistency with which players carry out these adjustments has been verified by measurement (Fletcher 1975) and agrees with the expectations of theory. Because blowing pressure is constrained in this way, players control the loudness of the sound by opening or closing the lip aperture, thus controlling the jet volume flow without affecting its speed.

As it stands, this theory is entirely linear. It is simple, however, to see the source of the nonlinearity. The jet, as already noted, has a bell-shaped velocity profile that can be described by the expression $V(y) = V_0 \operatorname{sech}^2(y/b)$. If the edge is displaced an amount y_0 from the jet centre-plane, then the flow intercepted by the edge when the jet deflection is y is

$$U_\mathrm{j}(y) = W\int_{-\infty}^{y_0+y} V_0 \operatorname{sech}^2(y/b)\,\mathrm{d}y = WbV_0\left[\tanh\left(\frac{y+y_0}{b}\right)+1\right] \tag{8.4}$$

where W is the jet width and $2b$ its thickness, as before. This flow is shown in figure 16, and exhibits a gentle nonlinearity, with saturation at both extremes as the jet flows entirely into or entirely outside the lip. This nonlinearity can be investigated (Fletcher and Douglas 1980)

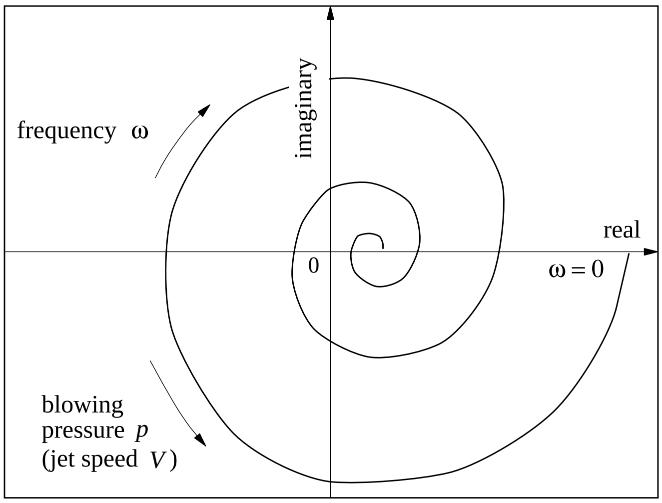

Figure 15. Qualitative behaviour of the complex acoustic admittance of an organ-pipe jet, as a function of frequency and blowing pressure. The jet acts as an acoustic generator only when the real part of the admittance Y_j is negative.

by assuming a simple sinusoidal jet deflection at frequency ω, which is a good approximation because of the frequency-selective behaviour of the jet amplification coefficient μ. We find that, if the offset y_0 is zero so that the jet motion is symmetrical about the lip, then only odd harmonics $(2n+1)\omega$ are generated. As the offset is increased, all harmonics $n\omega$ are generated, but particular harmonics are suppressed at particular offsets. For example 3ω is suppressed if $y_0 = 1.1b$ and 4ω if $y_0 = 1.7b$. These theoretical predictions are well borne out in experiment, and adjustments of the direction of the air jet are an important part of organ pipe voicing (Nolle 1979) as well as of flute performance technique.

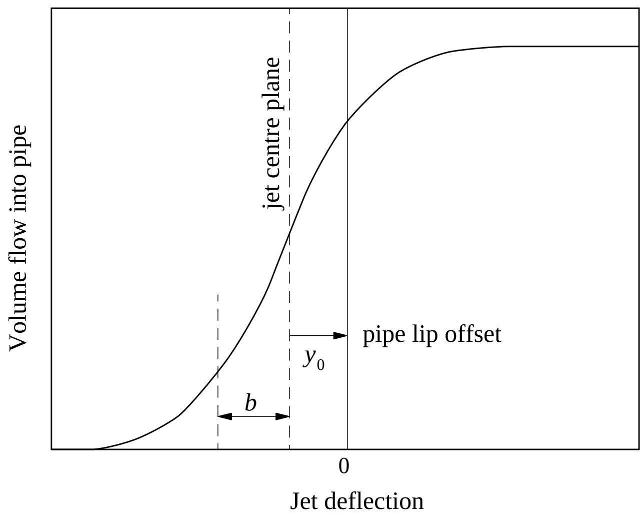

Figure 16. Jet volume flow into a pipe past a lip with offset y_0. The jet is assumed to have a bell-shaped profile with half-width b.

This mechanism provides considerable nonlinearity for the generation of harmonics which are then amplified by the resonator (or not amplified, in the case of even harmonics and a stopped cylindrical pipe). It also provides the mechanism for mode locking and, when the conditions are right, for multiphonic production. The onset transient of a typical organ pipe has already been presented in figure 3. Similar transients are characteristic of the sounds of other air-jet instruments, though they can be modified by the tonguing technique used.

The pipe organ is, of course, the largest and most complex of instruments, with thousands of pipes at a multitude of pitches all under the control of a single organist. The history of its development and the understanding of its design are fascinating subjects that would take us too far from the central theme of the present review. A more extended discussion, with extensive references, is given in Fletcher and Rossing (1998) ch 17.

9. Impulsively excited instruments

In an impulsively excited instrument, energy is transferred to the vibrating element in a time that is very short compared with the decay time of the vibration, and so the duration of the note. It is true that there is scope for nonlinear behaviour in the impulse itself, as will be discussed in relation to the piano, but most of the nonlinearity with which we are concerned is associated with the vibrating element itself. The system diagram of figure 1 can be modified to apply to this class of instruments provided the generator is regarded as impulsive and is disconnected after it has done its work.

We have described impulsively excited instruments as being 'incidentally nonlinear' and some of them, particularly struck or plucked strings, are scarcely nonlinear at all. The resonance frequencies of stretched strings are also quite closely harmonic, so that little modification is required to the linear-harmonic model to encompass their behaviour. The characteristic of harmonicity is abandoned once we come to discuss bells and the simpler types of gongs, however, and the inherent nonlinearity associated with rather flat shells becomes a dominant feature of the behaviour of some oriental gongs and cymbals.

Before entering this discussion, it is interesting to examine the role of geometry and elasticity in determining harmonicity in a broad sense. This can be derived from consideration of the differential equation governing the vibration in each case. For a string the equation for transverse wave propagation is

$$\rho S \frac{\partial^2 y}{\partial t^2} = \sigma S \frac{\partial^2 y}{\partial x^2} \tag{9.1}$$

where ρ is the material density, S the cross-sectional area and σ the tensile stress in the string. The wave velocity $c = (\sigma/\rho)^{1/2}$ is independent of frequency, and the fixed boundary conditions require a wave-number $k_n = \omega_n/c$ for mode n that is simply proportional to n. The mode frequencies ω_n are therefore harmonic, and the modal density $D(\omega)$ is constant.

For a thin bar, on the other hand, the transverse wave equation is

$$\rho S \frac{\partial^2 y}{\partial t^2} = -E S \kappa^2 \frac{\partial^4 y}{\partial x^4} \tag{9.2}$$

where E is Young's modulus and κ is the radius of gyration of the bar about its neutral section ($\kappa = r/2$ for a bar with circular cross-section of radius r). In this case the wave velocity is $c = (ES\kappa^2/\rho)^{1/4}\omega^{1/2}$ so that the mode frequencies increase roughly as n^2 (actually more nearly as $(n - \frac{1}{2})^2$ for a bar clamped at one end, and as $(n + \frac{1}{2})^2$ for a doubly clamped or doubly free bar). The modal density $D(\omega)$ is thus approximately proportional to $\omega^{-1/2}$ and the modes are widely spaced and generally not harmonically related unless the bar cross-section is systematically varied along its length.

For a square membrane, the equation (9.1) must be extended to two space dimensions. The mode frequencies are then proportional to $(n_1^2+n_2^2)^{1/2}$, and the modal density is proportional to the number of $(n_1,\ n_2)$ points between concentric circles in the $(n_1,\ n_2)$ plane. The result is a modal density $D(\omega)$ that is proportional to ω, so that the modes are progressively closer at high frequencies and cannot be harmonically related. Details of the modal frequency distribution depend upon the shape of the membrane, if it is not square, but the overall modal distribution at high frequencies does not.

For a square plate the same approach applies, but the factors $\omega^{1/2}$ in the modal density functions for a bar just cancel the extra factor ω contributed by the two-dimensional nature of the plate, so that the resulting mode density is independent of frequency. This is, however, only a statistical result, so that we cannot say that the mode frequencies are harmonic. The exact frequencies depend upon the plate shape and boundary conditions. It does suggest, however, that a properly shaped plate might have a significant number of its mode frequencies in approximately harmonic relationship. This same conclusion applies to the modes of plates that have been deformed into curved shells, and thus to the modes of gongs and bells, provided the material thickness is small relative to the local radius of curvature.

With this background, we now discuss the roles of inharmonicity and nonlinearity in various types of impulsively excited instruments. Strings have already been discussed as forming the primary oscillators of bowed-string instruments, and they are also the sound generators in a variety of plucked and hammered-string instruments. There is considerable similarity between the two classes, but we discuss them separately.

10. Plucked-string instruments

The family of plucked-string instruments—guitars, harps, zithers and harpsichords—has a long history. They have some features in common with the bowed-string instruments, the most important of which is the necessity of coupling the string through a bridge to a soundboard. The characteristics of that soundboard, and any associated structures, then modify the spectrum of the primary string vibrator during the radiation process.

Similar considerations apply to the bodies of guitars and lutes as have already been discussed in relation to violin bodies. The body must be adequately strong to resist string forces, but at the same time light enough that it can be set into a sufficiently large amplitude of vibration. Its modes must be well distributed in frequency, so that the radiated sound does not suffer undue changes in strength or tone quality from one note to another. These requirements have been met, in the case of guitars, by using a flat soundboard mounted on a hollow body with a vent hole. The flat top is stiffened by the application of numerous light ribs to its underside, and the pattern of these ribs is designed to achieve the desired distribution of resonances. There is much more variability between guitars by different makers in this respect than there is between violins.

Similar considerations apply to the harp, which has a long narrow soundboard, and to the harpsichord, in which the soundboard is quite large in order to reinforce adequately the lower notes of its compass. The harpsichord also has an angled lid that serves an important purpose in directing the sound horizontally towards the audience, rather than vertically. In most cases the instrument size is standardized, but harpsichords range from small spinets with oblique stringing and a soundboard less than 1 m in maximum dimension, to large concert harpsichords with two manuals, several courses of strings, some tuned an octave above or below the nominal pitch as in an organ, and a soundboard around 2 m in length. In both cases the musical compass is essentially the same, but the larger instrument has a richer and louder sound, as well as more variety of tone colour.

Much ingenuity has been employed over the centuries to develop the sound-radiating bodies of these instruments, particularly those such as guitars that are carried by the player. Recently, and following development of the new violin octet by Carleen Hutchins, Graham Caldersmith (1995) has designed and built an acoustically matched family of four guitar instruments, two smaller and one larger than the classical guitar. These are receiving a welcome from musicians (Guitar Trek 1990), and present less transition difficulty for players than do the violin family instruments.

From our present perspective, major interest centres on the plucked string and its behaviour. The elementary theory is well known, and this should be supplemented by including the stiffness so that the transverse vibration satisfies an equation of the form

$$\sigma S \frac{\partial^2 y}{\partial x^2} - \frac{ESr^2}{4}\frac{\partial^4 y}{\partial x^4} = \rho S \frac{\partial^2 y}{\partial t^2} \tag{10.1}$$

where ρ is the density, E the Young's modulus and $S = \pi r^2$ the cross-section area of the string, and σ is the tensile stress, so that $T = \sigma S$ is the string tension. This equation is still linear, but the normal modes, assuming fixed ends, are slightly stretched in frequency so that (Morse 1948, p 170)

$$\omega_n \approx n\omega_1 \left(1 + \frac{n^2\pi^3 E r^4}{8TL^2}\right). \tag{10.2}$$

This result can also be derived by substituting $g = -(ESr^2/4)(\partial^4 y/\partial x^4)$ in the equations (3.7). The inharmonicity is very small for the nylon strings of guitars, for which the Young's modulus is small, but can become appreciable for thick steel strings, as in the piano.

The next refinement is to recognize that the length of a vibrating string is greater than that of the same string at rest by an amount

$$\delta L = \int_0^L \left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{1/2} \mathrm{d}x \tag{10.3}$$

which increases the stress σ by an amount $E\delta L/L$. If $y(x, t)$ is expanded in terms of normal modes and substituted into (10.3), then the steady stress is increased by a factor

$$\delta\sigma = E\frac{\delta L}{\sigma L} = E\sum_n \frac{n^2\pi^2}{4L^2} a_n^2 \tag{10.4}$$

where a_n is the amplitude of mode n. In addition, if several modes are present, then there are sum and difference frequencies generated in the tension that may slightly change the frequencies of individual modes, as can be seen by adding such terms into the tension σ in equation (10.1). The behaviour is described by equations (3.6)–(3.8) if we take the nonlinear excitation function g to be $S\delta\sigma\partial^2 y/\partial x^2$. The overall result is that, if a string is plucked to very large amplitude, then the frequencies of all the modes will be slightly raised and will then fall back towards the small-amplitude value as the oscillation decays, giving an unpleasant 'twang'. Use of nylon strings in guitars and thin metal strings in harpsichords, in each case at high tension, minimizes this effect because either E or r is small.

There is one other refinement that should be mentioned, though it will not be pursued in detail, and that is the existence of longitudinal waves on the string. In the linear approximation, these are quite independent of the transverse waves, but in the correct nonlinear case the two sets are coupled because of the influence of transverse vibrations on string tension. A full account is given by Morse and Ingard (1968), and the simplified treatment above essentially relies upon the assumption that the velocity of longitudinal waves is sufficiently much greater than that of transverse waves on the string that the tension can be assumed to be constant along its length.

The sound spectrum produced by a plucked-string instrument depends upon the nature of the plectrum, the plucking point, and the resonances and radiating properties of the instrument body. The string is displaced by the plectrum into two straight segments joined by a short curved section, the shape of which is determined by the width and softness of the plectrum and the stiffness of the string. When an ideal string of length L is plucked with an ideal plectrum at a distance l from one end, the force $F = T(\mathrm{d}y/\mathrm{d}x)_{x=0}$ on the bridge has the form

$$F(t) = A \sum_n \left[\frac{\sin(n\pi l/L)}{n} \right] \cos n\omega_1 t \tag{10.5}$$

where ω_1 is the frequency of the first mode, as can be readily derived by carrying out a Fourier analysis of the initial string shape. Converted to a power spectrum, the envelope of this force is fairly constant out to about $\omega_n = (L/2l)\omega_1$ and then declines at 6 dB/octave for higher frequencies. Beneath this envelope the spectrum has zeros at the frequencies $(L/l)\omega_1$, and thus strictly missing modes if L/l is an integer, as shown in the examples in figure 17. This source spectrum is modified by the radiation characteristics of the instrument body, but clearly provides a means of tonal expression on plucked-string instruments, either built into the mechanism in harpsichords or under the fingers of the player in harps and guitars. Actually, although certain modes may be missing upon the initial excitation, nonlinear effects, particularly those at the bridge, are able to regenerate these modes, typically over a time of order 1 s (Legge and Fletcher 1984).

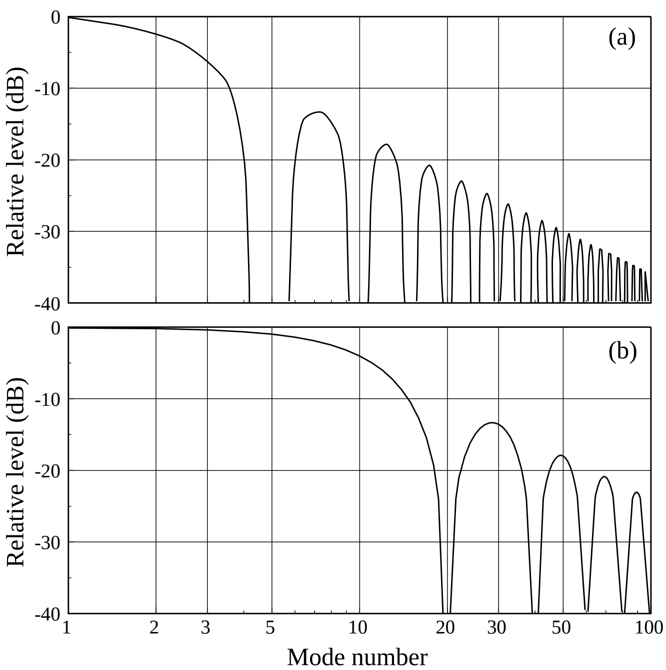

Figure 17. Envelope of the transverse force spectrum at the bridge for a string plucked at (a) one fifth and (b) one twentieth of its length from one end.

The decay of string oscillations is governed by a combination of internal losses in the string, losses to the soundboard through the bridge, and viscous losses to the surrounding air. The role of direct radiation damping of the string is negligible. Internal friction is caused mainly by heat conduction and grain-boundary movement in metals and by molecular rearrangement in plastics, neither of which gives very rapid damping. In multi-fibre strings made of materials such as silk or twisted gut, however, the internal damping is large, particularly at high frequencies, and the sound decays rapidly. The decay may even be linear in time, rather than exponential, if dry friction is dominant. An analysis of damping behaviour in the case of harpsichord strings has been given by Fletcher (1977).

Once a string has been set into oscillation, there can also be coupling between the two possible polarizations of the motion, brought about either by a direction-dependent compliance at the bridge or simply by coupling between the two polarizations through the agency of the longitudinal tension. This second effect occurs only when the two modes have appreciable initial amplitudes, so that the string motion is elliptical. In this case, the coupling causes rotation of the ellipse at an angular rate Ω given by

$$\Omega = A\left(\frac{Eab}{\sigma L^2}\right)\omega \tag{10.6}$$

where a and b are the amplitudes of the two polarizations, ω is the vibration frequency, σ is the tensile stress in the string, and A is a constant of order unity (Elliott 1980, Gough 1984, Valette 1995). This motion, which causes a quasi-periodic but gradually slowing variation in the normal component of force in the bridge, and thus in the sound intensity, adds interest to plucked-string sound, although other effects may overshadow it.

This treatment, so far, has considered only the behaviour of a single string. Interesting and important effects occur when there are two strings tuned to nominally the same frequency and passing over a common bridge, for the compliance of the bridge allows coupling between them, even in the linear approximation (Weinreich 1977, Gough 1981). There are two types of modes for a coupled system of two strings. In one, the strings move in co-phase with one another, and in the other they move in anti-phase. Coupling through the bridge will split the frequencies of these two modes by a small amount, and there will be some energy transfer between them, even for the case in which initially only the co-phase mode is excited. There is a clear audible effect, for the co-phase mode exerts a large oscillating force on the bridge, while the force exerted by the anti-phase mode is small. If the major loss of energy from the strings is through the bridge, then the co-phase mode will couple and radiate strongly but decay rather rapidly, while the anti-phase mode will radiate weakly and decay slowly. This effect depends sensitively upon the exact tuning of the two strings of the pair, but does make a considerable contribution to the sound of double-strung instruments, which exhibit a rapid initial sound decay followed by a long sustain.

It is appropriate, finally, to mention the Indian sitar, in which the strings pass over a curved bridge in such a way that it imposes a one-sided and amplitude-dependent constraint on the string motion, producing the characteristic sound of this instrument. The analysis is complex (Burridge et al. 1982, Valette 1995), as might be expected from the sound, and must be performed in the time domain, rather like discussion of the Helmholtz motion of a bowed string.

11. Hammered-string instruments

The piano is the main instrument to be considered in this section, although the clavichord (Thwaites and Fletcher 1981) and the hammered dulcimer (Peterson 1995) might also be mentioned. The principles of construction of the sound-radiating body are similar in each case, though the piano has an open soundboard while the other two instruments have a soundboard that forms the top of a vented box. The construction of the modern piano is very heavy, so that it can bear high string tension on its iron frame and consequently produce a loud sound when it is struck vigorously. The piano also has multiple stringing for all but its very lowest notes, so that its tone has a two-stage decay, as discussed in the previous section.

Despite the high string tension, the fact that the piano has steel strings of quite large diameter results in appreciable inharmonicity of the string resonances, particularly in the extreme bass and treble ranges. This is reflected in the tuning (Schuck and Young 1943, Martin and Ward 1961), for the piano has stretched octaves at both extremes of its compass, the total stretch over the keyboard being nearly one semitone. Apart from giving a slightly bell-like character to the extreme treble sound, this inharmonicity is hardly noticeable.

The behaviour of the strings and soundboard in a piano is very nearly linear, for only relatively small deflections of the heavy strings are required to produce a loud sound, and indeed the string tension is so high that this is all that can be achieved by even a forceful blow. Interest centres rather upon the interaction of the hammer with the strings.

The piano mechanism, as it has developed over nearly 300 years, is a marvel of precise design. The felt-coated hammer is brought smartly into contact with the string, at a velocity that is under the precise control of the player, and then rebounds to be caught safely clear. The fact that piano tone has a subtle gradation from gentleness to assertiveness results from the fact that the felt covering of the hammer is graded in compaction, and in any case becomes harder as it is compressed during the impact with the string. This impact occupies a time that is long compared with the travel time of waves to the close end of the string, and comparable with the time for wave travel to the remote end. Discussion of the details of this impact and of the subsequent motion of the string are therefore complex, and best carried out in the time domain (Hall 1992, and previous papers cited therein).

Because nonlinearity is not really a feature of piano physics, except for the initial hammer contact, and because the instrument is so highly developed as a piece of mechanism, it will not be considered further here. Those who wish to follow details of its construction and operation should consult the discussion and references listed in Fletcher and Rossing (1998) ch 12.

12. Drums

An elementary exposition of drum behaviour usually follows that of strings, for the restoring force is primarily that of tension, with membrane stiffness as only a minor consideration. The drum membrane is, however, two-dimensional, so that the frequencies of its vibrational modes become progressively more closely spaced with increasing mode number—indeed the modal density $D(\omega)$ increases in proportion to ω, as was discussed above. This means that there is rather little tonal quality to the sound of a drum, and any that is produced must derive from the lowest few modes.

Timpani are the most common form of orchestral tuned drums, and the mode that is responsible for the tone is the first antisymmetric mode (1,1), with angular dependence $\sin\phi$ and one nodal diameter. This is the mode selected, since, being dipolar when one side is enclosed by the kettle, it radiates energy relatively slowly and produces a prolonged sound, while the circularly symmetric lowest mode is a monopole and is very rapidly damped. This is evident if a player strikes the drum in the centre instead of near the edge. The shape and size of the drum kettle support this antisymmetric mode, and the associated internal air motion might be termed tidal—it does not involve air compression but simply adds air mass to the motion of the membrane (Rossing 1982, Christian *et al* 1984). There is also a substantial mass loading from the air outside the kettle. A properly designed kettle produces approximately the frequency ratios $1:1.5:2:2.5$ for the (1,1), (2,1), (3,1) and (4,1) modes, which have respectively 1, 2, 3 and 4 nodal diameters and no nodal circles. Although the other low-frequency modes have incommensurate frequencies, this series produces a strong tonal impression with a perceived pitch an octave below that of the (1,1) mode. Because the kettle contributes simply an air loading and not a resonance, the drum can be tuned over a substantial range simply by changing the membrane tension, without destroying the frequency relationship between these modes.

Another interesting class of drums has a membrane that is centrally loaded by a heavy patch. Principal among these drums are the Indian tabla, which looks like a small kettledrum, and the mrdanga, which is a barrel-shaped drum with two end membranes. Raman (1934) was among the first to study these drums, and showed that they possess as many as five modes in approximately harmonic relationship, giving a strong pitch impression. This design has been taken to its logical refinement with the development by Aebisher and Gottlieb (1990) of an annular kettledrum with a very large number of harmonically related modes.

It is interesting to note that many drums have two membrane heads that are generally tuned to the same note, or at least to approximately the same tension. This doubles the number of vibrational modes of the combined drum, with each membrane mode effectively splitting into a pair in which the displacements are symmetric between the two membranes and another pair in which the displacements are antisymmetric. Because the symmetric mode compresses the enclosed air, its frequency is higher than that of the antisymmetric mode, in which the air simply moves as a mass load. The total picture is actually more complex than this because of other (linear) interactions between modes, but the final result is greater radiation of sound, particularly from the symmetric modes.

Although all these drums could in principle be struck hard enough to produce a pitch-glide nonlinearity of the type discussed for plucked strings, this does not appear to happen in musical practice. There is, however, one type of drum in which nonlinearity plays a large part, and that is the military snare drum. In this instrument, which has two membrane heads, a set of strands of wire or gut is stretched across the lower (unbeaten) head. For light strokes on the upper head, the snares remain in contact with the lower head and simply reduce its motion. For heavy blows, however, the lower head lifts off the snares and then recontacts them once in each cycle, the impact of the snares on the head generating a wealth of high-frequency oscillations and a characteristic unpitched buzz or rattle sound.

13. Bells, gongs and cymbals

The distinction between bells, gongs and cymbals derives both from their shape and from the way in which they are played. Bells are deep cup-like structures of rather heavy metal that are usually supported from the centre of their top surface and struck on one side. The hammer or clapper may be of wood, as in some Eastern bells, but is usually of metal. The vibrational modes of the bell that are of importance are therefore those with one or more nodal lines running through the central support boss. A gong is generally a much shallower and lighter circular dish with a flat or slightly curved central section, and is usually supported by a light cord passing through two holes near its edge. It is ordinarily struck in the centre of the central section using a soft padded hammer, so that it is primarily circularly symmetric modes with no nodal diameters that are excited. A cymbal is generally a shallow uniformly curved dish of thin metal with some sort of central boss from which it is supported. It can be struck near the edge with a hard stick, or two cymbals can be brushed together, making contact at their edges. The modes excited are presumably largely those with nodal diameters passing through the central support. Examples of the profiles of these instruments are shown in figure 18.

Although bells, gongs and cymbals are not a primary part of the Western musical tradition, they do provide accents in many compositions, and bells of course have a long ecclesiastical

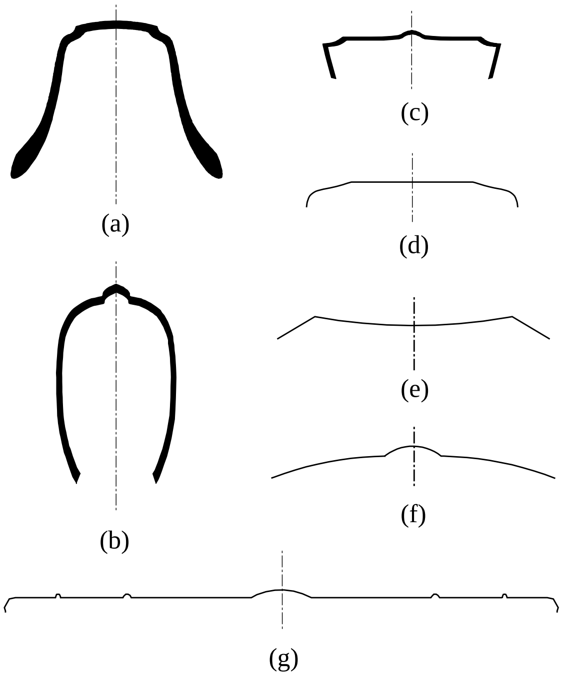

Figure 18. Profiles of representative members of the bell, gong and cymbal family. (a) English church bell, (b) Japanese temple bell, (c) Indonesian gamelan gong, (d) Chinese opera gong, (e) Turkish cymbal, (f) orchestral cymbal and (g) Chinese tam-tam. (Not all to the same scale.)

history. Some Eastern musical traditions, however, such as the Indonesian gamelan, are based upon the sounds of gongs and other percussion instruments, and consequently have a set of consonances and a musical scale that are different from those of Western music (Sethares 1998, ch 8).

Within this group of instruments, collectively called idiophones since there is just one resonant vibrating and radiating element for each note, a few have their modes adjusted so that those that are most prominent are in nearly harmonic relationship. One example is the set of tuned wooden bar instruments of the marimba and xylophone families. The normal modes of a simple uniform bar with both ends free are quite closely described by a series $f_n \approx A(n+\frac{1}{2})^2$ with A a constant determined by the bar dimensions, density and elastic modulus. This sequence is $1.0:2.76:5.41:8.93\ldots$, which is very far from being harmonic. Shaping the bar profile by cutting a parabolic arch into its underside, however, allows the second mode to be tuned to either 3 or 4 times the frequency of the fundamental (Orduña-Bustamente 1991, Bork 1995), and these modes can then be reinforced by the nearby presence of a tubular resonator. The bars of these instruments are quite stiff, and there is no significant nonlinearity in their vibrations.

13.1. Bells

Western church bells are another type of idiophone in which a tuning process is used to give closely harmonic modes. In the case of bells, the general pattern of mode frequencies is fixed by the cross-section as cast, which is of the form shown in figure 18(a). There is scope, however, for minor adjustment by using a vertical lathe to turn metal off the inside of the bell at specific places (Perrin *et al* 1983, Rossing and Perrin 1987). The target frequencies for the first few modes are in the ratios 0.5 : 1.0 : 1.2 : 1.5 : 2.0 . . . and these can be quite closely matched. The first mode, called the hum, is not prominent, and the perceived pitch is usually that of the second mode or 'prime', perhaps because it is reinforced by the harmonically-related modes with relative frequencies 2, 3 and 4. The tone is complex, however, particularly because of the presence of the minor-third interval 1.2. The sound of bells playing harmony is therefore, to say the least, 'characteristic'. Lehr (1987) has designed and built a set of major-third bells with smooth, but for this very reason marginally non-bell-like, tonal quality.

The vibration of bells is essentially linear because of their great wall stiffness, but there is nevertheless one nonlinear aspect of the produced sound. This arises because the first mode, which is responsible for the hum tone, has a shape in which the axial cross-section of the bell oscillates between two ellipses oriented at right angles to each other. Since the area enclosed by an ellipse is smaller than that enclosed by a circle of the same perimeter, this oscillation at frequency f_1 moves air in and out of the bell mouth at frequency $2f_1$ and generates a tone that reinforces the prime.

One other feature of bell vibration is worthy of mention, and that is that all the modes noted above are actually doubly degenerate because of the rotational symmetry of the bell. In practice this symmetry is unlikely to be exact, so the degeneracy is lifted and each mode actually consists of a closely spaced doublet, and these generally produce very slow beats that are clearly audible.

There are, of course, bells built in traditions other than this, and both Eastern church bells and the large temple bells of China and Japan follow quite different design practices, a typical profile being shown in figure 18(b). Some ancient Chinese bells have been discovered that dispense with circular symmetry in favour of a slightly pointed elliptical cross-section, with the result that the two modes that are degenerate in a circular-section bell now have very different frequencies. These modes can be separately excited by striking the bell in different places, so that each bell can produce two different pitches (Rossing *et al* 1988).

13.2. Gongs

The normal modes of gongs are similarly inharmonic, but no attempt is made to tune them to a harmonic or quasi-harmonic series. Instead, the shape of the gong determines its spectrum and, when gongs are used in combination, and with the addition of tuned-bar instruments as in the Indonesian gamelan orchestra, their modal frequencies in turn determine the musical scale. The subject is too complex to discuss here, and the interested reader is referred to Sethares (1998) ch 8 for more information and references.

The gongs of the gamelan do not exhibit any marked nonlinearity, because they have thick and highly curved walls as shown in figure 18(c), and are played at a moderate dynamic level. The smaller and flatter gongs of the Chinese opera orchestra, as shown in figure 18(d), however, are made of bronze no more than 2 mm in thickness and, since they represent climaxes in the action, are struck very vigorously with a padded hammer. The nearly flat central section of the gong is the active vibration region, while the stiffer conical surround both supports this disk and acts as a baffle to increase the sound radiation. There are actually two superficially similar gongs, but one has a quite flat central region while that of the other smaller gong is gently domed—only about 1 mm over its 100 mm diameter central region. This slight difference in geometry makes an immense difference to the sound in loud playing, though if the gongs are stuck softly they produce a simple and nearly sinusoidal tone at the fundamental frequency of the primary mode.

In the case of the strictly flat gong, the central vibrating region behaves like a circular plate rigidly clamped at its boundary. The gong is struck with a padded stick very close to its centre, so that most of the energy is transferred to the circularly symmetric fundamental mode. For large excitation of this mode, the material of the gong is stretched, just as for the oscillations of a string, and the frequency is higher than that for the linear small-amplitude excitation, as shown in the upper curve of figure 19. The initial strike typically produces a mode amplitude of more than 1 mm and gives an increase in pitch of as much as 20 per cent, or a minor third, at the initiation of the sound. As the vibration decays over a time of order one second, the pitch glides downwards to its small-amplitude value, producing a very characteristic sound.

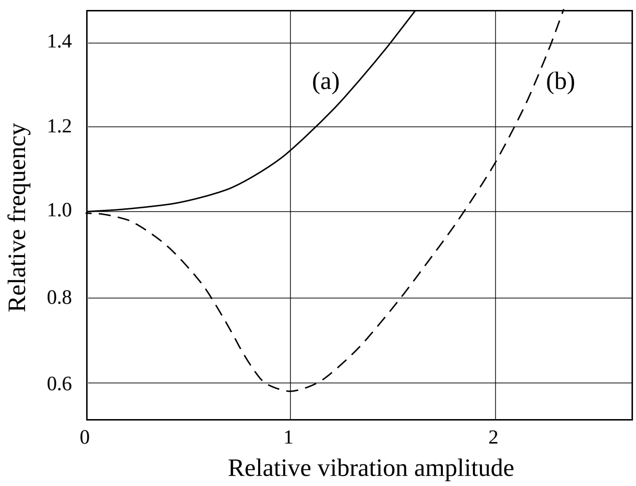

Figure 19. Frequency as a function of mode amplitude for (a) a Chinese opera gong with a flat central section, and (b) a similar gong with a slightly domed central section. In case (a) the amplitude is scaled in terms of the thickness of the gong material, while in case (b) it is scaled in terms of the dome height for a material thickness that is 0.4 times the dome height.

In the case of the smaller gong with its slightly domed surface, the effect of the elastic distortion associated with vibration in the first mode is quite different—for displacement upwards the material of the gong is placed under tension, while for movement downwards it is under compression until the displacement exceeds twice the dome height, when the stress becomes tensile again. The behaviour of the mode frequency with amplitude can be calculated (Fletcher 1985) and has the form shown in the lower curve of figure 19. In practice the mode amplitude achieved does not extend as far as the minimum in the frequency curve, with the result that the initial pitch is flat, again by as much as 20 per cent or a minor third, and glides upwards towards the small-signal value as the sound decays over a second or so. This produces a striking contrast with the sound of the downward-gliding gong.

13.3. The tam-tam

The large flat Chinese tam-tam makes its appearance in the Western orchestra and is used just occasionally for impressive accents. As shown in the cross-section of figure 18(g), it is circular, almost a metre in diameter, and made from quite thin bronze only 1 to 2 mm in thickness. This sheet is shaped to a large central dome, which is where the gong is struck using a heavy and softly padded striker, and has three or four rings of small hammered bumps, each ring containing 50 to 100 small domes perhaps 15 mm in diameter and 8 mm in height.

When the gong is hit, the prime excitation is of the circularly symmetrical fundamental mode, which typically has a frequency around 80 Hz and gives a massive deeply pitched thud. Over a time of 2 to 3 seconds, however, the sound changes to a high-frequency non-tonal shimmer which finally dies away. Examination of the vibration of the gong in this later stage reveals that the centre is essentially motionless, and the vibration is confined to the outer third of the area.

The vibrational behaviour of the tam-tam and related gongs has been investigated by Rossing and Fletcher (1981) and in more detail by Legge and Fletcher (1989). The radiated sound in both early and late stages is shown in figure 20, from which it is clear that there is actually a transfer of vibrational energy from low-frequency to high-frequency modes over time. This behaviour is also clear from Sonagraph records of the sound of the instrument.

The mechanism by which this occurs has not been completely elucidated, but one contributing factor appears to be mechanical nonlinearity at the rings of raised bumps. Excitation of the high-frequencies in this way would also break the initial circular symmetry of the vibration and pass energy to modes with an angular dependence of order equal to the number of bumps in the ring concerned. These angularly dependent modes would naturally be confined to the periphery of the gong, in the same way as similar modes on a drumhead or on a circular plate, which look rather like $r^m \sin m\phi$.

The frequency multiplication mechanism was elucidated in an earlier paper by Legge and Fletcher (1987) for the case of symmetrically kinked bars. Essentially what happens is that a large-amplitude vibration of frequency f generates a tension force of frequency $2f$ as discussed in section 10. At a sharp change in surface slope, this tension force excites a transverse vibration of frequency $2f$ in the adjoining section of material, the exciting force being of second order and thus proportional to the square of the initial mode amplitude. The $2f$ vibration takes a time of order 0.1 s to build up to full amplitude. The tension force at frequency $2f$ is applied to the second section of surface at an angle that itself varies at a frequency f, and this therefore also generates a transverse vibration at frequency $3f$, the interaction being of third order. This cascade of energy to higher frequencies in a tam-tam involves several steps and takes a time of order 1 s. Meanwhile, the low-frequency fundamental mode is heavily damped by radiation and dies out rather rapidly, leaving the energy distributed over the higher modes which do not radiate so rapidly.

While this description, supplemented by equations, appears to explain the general features of the sound of a tam-tam, there are indications that it is really more complex than this, as will be discussed in the next section.

13.4. Cymbals

The profile of a typical orchestral cymbal is shown in figure 18(f). The bronze from which it is normally made is only about 1 mm thick, but curvature stiffens the shell and, in addition, there are usually fine concentric ridges over the whole surface that are imparted during the manufacturing process.

Nonlinearity is very much a feature of normal cymbal sound, though it is possible to excite ordinary normal modes on a cymbal if the amplitude is small (Rossing and Peterson 1982, Rossing and Shepherd 1983, see also Fletcher and Rossing 1998 ch 20). The sound spectrum, however, appears to be much more complex than this, and the indications are that the vibration is actually chaotic at normal amplitudes, and also shows the same sort of energy cascade to higher frequencies found in the tam-tam.

Normal excitation of a cymbal by a glancing blow near the edge will excite many angularly dependent modes, so that the behaviour is not easy to study. It has been found, however, that both flat circular plates and actual cymbals, including the large heavy Turkish cymbal shown

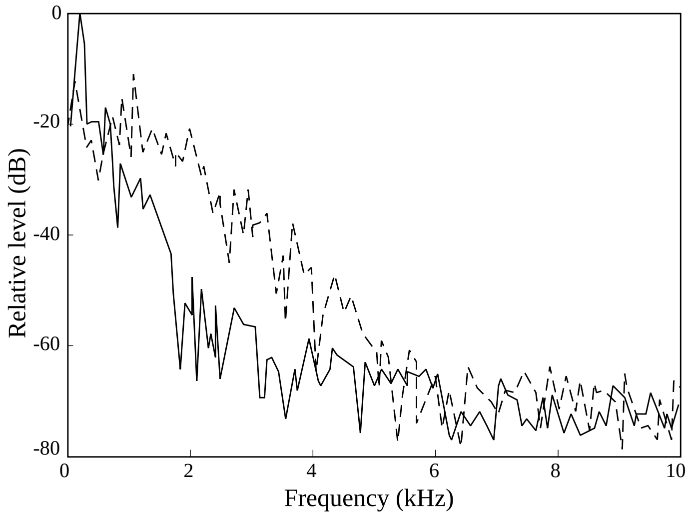

Figure 20. Spectrum of the sound radiated by a large tam-tam excited by a central blow with a soft striker immediately after the initial impact (full line), and after a time of about 3 s (broken line). The reference level is the same in each case. Note the transfer of energy from low-frequency to high-frequency modes (Legge and Fletcher 1989).

in figure 18(e), exhibit complex behaviour even when excited simply and sinusoidally at their centre (Legge and Fletcher 1989, Fletcher *et al* 1990). Patterns that have been observed include generation of subharmonics of orders 2, 3 or 5, and a transition to what appears to be a chaotic vibration that sounds very much like the normal large-amplitude vibration of a cymbal. A beginning has been made on the analysis of these centrally-excited chaotic vibrations by Touzé et al. (1998), who show that the number of active degrees of freedom in the particular vibration they analysed was between 3 and 7. No one has yet succeeded in analysing the sound of a normally struck cymbal at this level of detail.

14. Conclusion

This survey has examined the most important aspects of nonlinear behaviour in a wide variety of instruments. In the case of sustained-tone instruments, nonlinearity is essential to weld the sound produced into the coherent complex harmonic tone that is the basis of all Western music. Without it our musical heritage would be very different from what it is. Against this harmonic background, inharmonic instruments such as drums, bells and cymbals provide flashes of orchestral colour, and sometimes the chaotic crash of a cymbal or tam-tam.

How different is the music of some Eastern cultures that is based upon the linear but inharmonic sounds of softly played gongs and metal bars. Here it is the nonlinear harmonic sound of the flute or human voice that provides occasional tonal contrast!

Of course, with modern computer technology it is possible to synthesise any sound at all, but certain rules must be followed if anything of artistic worth is to be produced. Many of these rules relate to harmony and concordant scales and have now been admirably elucidated by the work of Sethares (1998). But these rules can be applied to the sounds produced according to any conceivable spectral recipe (even the x-ray diffraction spectrum of morphine!). To bring order to the huge range of possibilities, it is essential to understand the sounds of real instruments, even if only to then synthesize sounds that they could not possibly produce!

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