Jet-drive mechanism in organ pipes*

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The analysis given by Elder [J. Acoust. Soc. Am. 54, 1554 (1973)] of the mechanism of interaction between a time-varying air jet flow velocity and a resonant organ pipe is extended to deal with the case in which a jet of fixed flow velocity is deflected to cut a sharp pipe lip. Similar terms arise in the analysis but the nonlinearities are less important than in Elder's case. The generation of higher transverse pipe modes by the jet asymmetry is considered and it is concluded that such modes may play a significant part in coupling ordinary pipe modes to pipe-wall vibrations.

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INTRODUCTION

In an important recent paper,[1] Elder has analyzed the way in which a fluctuating air jet can interact with a resonant air column to provide a driving mechanism for sound generation, as in organ flue pipes. His treatment encompasses the driving interactions discussed earlier by Helmholtz, Rayleigh, Cremer, and Coltman and puts these into perspective in a more complete theory.

Elder's model is, however, itself one extreme case of a still more general situation, in that he assumes the pipe to be driven by an air jet whose velocity varies while its cross section remains constant. In a real situation, as we see presently, both the velocity and the cross section of the jet may vary and this leads to additional complications. It is the purpose of this paper first to set up the equations for the general case and then to investigate the opposite extreme in which the air jet cross section varies while its velocity remains constant. We find that the driving mechanism is much more nearly linear in this extreme than in that discussed by Elder.

Finally we give some consideration to the relation between these idealized models and the situation in a real pipe. We find that a variety of new effects may enter which have a significant influence on pipe speech.

I. JET MOMENTUM EQUATIONS

Elder considers a jet with cross section S_j and velocity U_j interacting with a resonant combination of a pipe of cross section S_p and a pipe-mouth of cross section S_m, the velocities in these being U_p and U_m, respectively. Assuming the jet to transfer momentum to the fluid in the pipe over a mixing length Δx, he then applies considerations of momentum flow to relate the various velocities and pressures involved. There is, in fact, a hidden assumption in this development in that the walls of the pipe are assumed to exert no component of force along the pipe axis, which can only be true if the total cross-section remains constant so that

$$S_p = S_m + S_j . \quad (1)$$

The idealized physical situation is therefore as shown in Fig. 1, which differs slightly from Elder's arrangement, as will be discussed below.

In a real organ pipe, and in our idealized situation, the jet is formed by a flue at some distance upstream from the mouth plane M. It is therefore reasonable to idealize the mouth velocity distribution U_m to be uniform over the whole pipe cross section so that the actual velocity over the jet area S_j is not U_j but $U_j + U_m$. This minor modification to the model, which is more appropriate to our geometry than to the confined jet assumed by Elder, produces a great simplification in the resulting equations.

We now make use of (1) and of the continuity equation (assuming $\Delta x \ll \lambda$), which can be written

$$S_p U_p = S_m U_m + S_j (U_m + U_j) . \quad (2)$$

Generalizing Elder's analysis so that both U_j and S_j may vary with time, we find, after a little algebra, in place of his Eq. (10),

$$\rho \Delta x (dU_p/dt) + (p_p - p_m) = \rho A_{jp}(1 - A_{jp}) U_j^2 , \quad (3)$$

where ρ is the density of air, p_p and p_m are the pressures at the pipe and mouth surfaces P and M of Δx, respectively, and the A_{ip} are areal ratios defined by

$$A_{ip} = S_i / S_p . \quad (4)$$

Following Elder we now resolve p_p and p_m and the velocities U_p, U_m, and U_j into Fourier components at frequencies $n\omega$. The relations between these can be written in phasor notation, as

$$\mathbf{p}_m = p_{m0} - \sum_{n=1} \mathbf{Z}_{mn} \mathbf{U}_{mn} S_p , \quad (5)$$

$$\mathbf{p}_p = p_{p0} + \sum_{n=1} \mathbf{Z}'_{pn} \mathbf{U}_{pn} S_p , \quad (6)$$

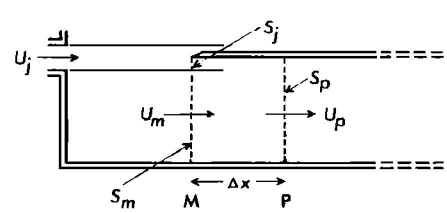

FIG. 1. Idealized model for the interaction of an air jet of velocity U_j with a resonant pipe of cross section S_p. An area S_j of the jet cross section enters the pipe in the mouth-plane M. The jet exchanges momentum with air in the pipe over a mixing-length Δx which is short compared with the sound wavelength involved.

where $\mathbf{Z}_{mn}$ and $\mathbf{Z}'_{pn}$ are the values of the complex acoustic impedances at the nth harmonic evaluated along the axis of the pipe looking outwards from the control volume. Because the velocity component $\mathbf{U}_{mn}$ is uniform across the whole pipe cross section, S_p occurs in (5) rather than S_m as in Elder's corresponding equation.

Because Eq. (3) is quite general we could now choose either Elder's extreme case in which the area S_j is held constant and the jet flow varied through variation of U_j, or the opposite extreme in which U_j is constant and the jet flow is varied through variation of S_j. This latter would correspond to the situation in a real organ pipe if the velocity profile across the jet were of "top-hat" form. We shall not take up the intermediate case in which both S_j and U_j vary.

In most real organ-pipe situations $A_{jp} \ll 1$, corresponding to a narrow jet entering a wide pipe, and an examination of (3) immediately tells us something about relative nonlinearity in the two extreme cases. The nonlinearity is, in fact, confined to the right-hand side of (3) and is much more important for varying U_j than for varying S_j (or A_{jp}) if $A_{jp} \ll 1$. The nonlinearity in the two cases is comparable if $A_{jp} \simeq 0.5$.

Our Eq. (3) has fewer nonlinear terms than the corresponding equation in Elder's development in any case because of our geometrical assumptions. Much of Elder's nonlinearity arises from the change in effective duct cross section between the pipe and mouth regions. In our version this particular form of nonlinearity is disassociated from the pipe-drive problem and is considered later in connection with the nonlinearity of the external pipe mouth.

II. JET-DRIVE EQUATIONS

It is now a simple matter to substitute (2), (4), (5), and (6) into (3) to derive an expression for the acoustic flow in the pipe. This is a simpler expression than that for the mouth flow as derived by Elder. If we take U_j to be constant and the relative jet area A_{jp} to be modulated by a whole series of Fourier components $\mathbf{A}_{jpn}$, then the resulting pipe acoustic flow $\mathbf{Q}_{pn} = \mathbf{U}_{pn} S_p$ at frequency $n\omega$ is

$$\mathbf{Q}_{pn} = (\mathbf{Q}_{pn})_{\mathrm{I}} + (\mathbf{Q}_{pn})_{\mathrm{II}} + (\mathbf{Q}_{pn})_{\mathrm{III}} \,, \tag{7}$$

where

$$(\mathbf{Q}_{pn})_{\mathrm{I}} = [\mathbf{Z}_{mn}/(\mathbf{Z}_{pn} + \mathbf{Z}_{mn})] S_p U_j \mathbf{A}_{jpn} \,, \tag{8}$$

$$(\mathbf{Q}_{pn})_{\mathrm{II}} = [\rho U_j^2/(\mathbf{Z}_{pn} + \mathbf{Z}_{mn})] \mathbf{A}_{jpn} \,, \tag{9}$$

$$(\mathbf{Q}_{pn})_{\mathrm{III}} = -[\rho U_j^2/(\mathbf{Z}_{pn} + \mathbf{Z}_{mn})] \sum_{n'} \mathbf{A}_{jpn'} \mathbf{A}_{jp(n-n')} \,, \tag{10}$$

and $\mathbf{Z}_{pn}$ is the pipe impedence evaluated now at plane M rather than plane P

$$\mathbf{Z}_{pn} = \mathbf{Z}'_{pn} + j\rho n\omega \Delta x / S_p \,. \tag{11}$$

The three terms in (7) are similar to those found by Elder. $(\mathbf{Q}_{pn})_{\mathrm{I}}$ is a volume-induced flow like that described by Cremer and Ising, $(\mathbf{Q}_{pn})_{\mathrm{II}}$ is a pressure-induced flow like that treated by Coltman, and $(\mathbf{Q}_{pn})_{\mathrm{III}}$ is a new nonlinear flow. In our physical model, in contrast to that of Elder, all the nonlinearity is concentrated into $(\mathbf{Q}_{pn})_{\mathrm{III}}$ and the linear terms $(\mathbf{Q}_{pn})_{\mathrm{I}}$ and $(\mathbf{Q}_{pn})_{\mathrm{II}}$ have simpler form. Clearly the nonlinearity tends to vanish, for a given jet volume flow, as the jet flow velocity is increased and its cross section decreased.

As Elder has noted, the mouth impedence near resonance is given approximately by

$$|\mathbf{Z}_m(\omega)| \simeq \rho\omega \Delta L / S_p \,, \tag{12}$$

where ΔL is the end-correction length at the pipe mouth. The relative importance of the first two terms in (7) is therefore given by the ratio

$$|(\mathbf{Q}_{pn})_{\mathrm{I}}/(\mathbf{Q}_{pn})_{\mathrm{II}}| \simeq n\omega \Delta L / S_p U_j \tag{13}$$

which, in many practical situations, is usually not far from unity.

A treatment similar to this could be given for Elder's case in which U_j varies, or for the more general case in which both U_j and A_{jp} vary. The linear terms are essentially the same in all cases and only $(\mathbf{Q}_{pn})_{\mathrm{III}}$ changes.

III. OTHER NONLINEARITIES

The results of our analysis are simpler than Elder's because we have not considered any nonlinearity arising from change in duct cross section near the pipe mouth. Any such change in dimensions or flow direction inevitably produces nonlinearity and mode coupling because of the relatively large acoustic velocities involved and the basic nonlinearity of the equations for wave propagation. A complete theory must include such effects but, except at extreme acoustic levels, their influence on the behavior of the pipe is likely to be small in comparison with other nonlinearities now to be discussed.

In a real organ pipe or flute the exciting jet generally has a bell-shaped velocity profile and sweeps across the pipe lip from a position well outside to a position well inside the pipe. Even if the deflection of such a switched jet is simply sinusoidal, its flow into the pipe will give a driving force rich in harmonics. This will excite the relevant pipe modes which will in turn interact with the jet in a way which has been described elsewhere.[2] In the presence of such extreme nonlinearity the contribution of terms like $(\mathbf{Q}_{pn})_{\mathrm{III}}$ or of the duct cross-section change at the pipe mouth will be overshadowed and can usually be ignored.

IV. HIGHER PIPE MODES

It is customary in most discussions of the behavior of wind instruments to consider only the zeroth-order transverse mode of the pipe, for which the wave fronts are planes normal to the pipe axis. An exception is the work of Benade and Jansson[3] on rapidly flaring horns, in which consideration is given to the second axially symmetric mode (which they call a p wave, a somewhat unfortunate title since by quantum-mechanical analogy it should properly be called a $2s$ wave).

In the case of musical instruments it is generally a good approximation to neglect all the modes with nonzero transverse wave number because, with common pipe dimensions and sounding frequencies, these modes have imaginary phase velocities and are very strongly attenuated. It is evident, however, that the air jet exciting an ordinary organ pipe does so in a manner which is very far from being uniform over the pipe cross section, so that we should expect it to couple strongly to some of the higher pipe modes.

The standard analysis of modes in a cylindrical pipe, as given for example by Morse and Ingard,[4] shows that the pressure associated with the (m, n)th transverse mode can be expressed as

$$p_{kmn} = \psi_{mn}(r, \phi)\exp[j(kz - \omega t)], \tag{14}$$

where k is the axial wave number,

$$k^2 + \kappa_{mn}^2 = (\omega/c)^2, \tag{15}$$

and

$$\psi_{mn} = \begin{matrix}\cos\\ \sin\end{matrix}(m\phi)J_m(\kappa_{mn} r). \tag{16}$$

For a pipe with rigid walls, the transverse eigenvalues are determined by the condition that $\partial\psi_{mn}/\partial r = 0$ at the pipe wall $r = R$, and this condition is modified slightly for other wall boundary conditions. If $\omega < \kappa_{mn} c$ then the (m, n) mode will not propagate but is attenuated along the pipe axis. If we write $\kappa_{mn} = \pi\alpha_{mn}/R$ then the α_{mn} increase numerically in the order $\alpha_{00}, \alpha_{10}, \alpha_{20}, \alpha_{01}, \ldots,$ so that, after the fundamental (0, 0), the (1, 0) mode [which might properly be called a p-wave from its angular dependence (16)] has the lowest cutoff frequency, followed by the d-wave (2, 0).

In an organ pipe the mouth width is generally about one quarter of the pipe circumference so that it subtends an angle of about $\pi/2$ at the center and the jet should couple strongly to modes (m, n) with $m = 0, 1, 2$. The jet deflection inside the pipe is usually almost comparable with R so that modes with $n = 0$ or 1 will be most strongly coupled. We are therefore led to consider primarily the plane-wave mode (0, 0) and the first two higher modes (1, 0) and (2, 0). The wavelengths of the fundamental and the next few higher modes of a normal organ pipe are generally much greater than the pipe circumference $2\pi R$ and, since $\alpha_{10} \simeq 0.59$ and $\alpha_{20} \simeq 0.97$, Eq. (15) shows that for these two modes $k \simeq i\pi\alpha/R$. Both modes are therefore nonpropagating and the pressures along the pipe are all in phase but attenuate in amplitude by the factor e^{-1} in axial distances of about $R/2$ and $R/3$, respectively.

We now ask what effect, if any, the presence of these higher modes may have on the behavior of the jet/pipe system. Apart from any small effects due to viscosity or thermal loss to the walls, the modes interact near the pipe mouth because of the influence of the jet and of the geometrical disturbances caused by the pipe mouth. In the main pipe body, however, the higher modes are orthogonal to each other and to the plane-wave modes and, indeed, most of their influence has already been implicitly included through the concept of a mixing length Δx and a mouth impedance $\mathbf{Z}_m$. To first order then, inclusion of these higher modes does not affect the existing treatment.

For a pipe of reasonable length and with rigid walls there is little contribution to the impedance presented to the jet by radiation from higher modes, which in any case has a multipole character. Such modes therefore present essentially a mass reactance load of magnitude about[5]

$$\mathbf{Z}_{pmn} = \rho c[1 - (c\kappa_{mn}/\omega)^2]^{-1/2} \tag{17}$$

per unit area to the driving source. This is much less than ρc, except for ω near the mode cutoff frequency $c\kappa_{mn}$, and, since in a formal analysis $\mathbf{Z}_{pmn}$ would replace $\mathbf{Z}_{pn}$ in expressions like (8)–(10) for part of the flow due to this mode, the velocity amplitude of higher modes near the pipe mouth could be considerable. Because of the short attenuation distance for these higher modes, however, most of this complication can be encompassed in the mixing-length parameter Δx which does not enter the final results provided $\Delta x \ll \lambda$.

The one way in which these higher modes could be important seems to be through their interaction with nonrigid pipe walls. The plane-wave mode (0, 0) can affect the pipe walls only through a symmetrical change in the pipe radius R, and the admittance of the pipe walls for such a distortion is very small. The p-type (1, 0) mode, however, couples to the pipe by means of a transverse displacement of the whole pipe body near the mouth. Such a displacement has a much larger admittance, especially for a thin-walled pipe, and a complex behavior determined by the transverse modes of vibration of the pipe body as a whole. The d-type (2, 0) mode similarly can couple to elliptical distortions of the pipe crosssection near the mouth and the admittance for this process may be large in a thin-walled pipe. Controversy about the influence of wall vibrations on pipe sound quality, particularly in the transient regime, has not yet been conclusively resolved but it appears that the coupling chain (0, 0) mode → jet → $(m, 0)$ mode → pipe body has not hitherto been considered and may prove significant.

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