

## ON THE PERTURBATION SERIES FOR RADIATIVE EFFECTS

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**Abstract**—We present a generalization of the recently developed first-order perturbation approximation for the computation of radiative effects such as layer heating rates and surface fluxes. The basis for this generalization is Dyson's equation for Green's function or the inverse transport operation.

### INTRODUCTION

In two recent papers,<sup>1,2</sup> Box and his co-workers have developed a first-order perturbation approximation for the calculation of radiative effects such as surface fluxes and layer heating rates. Preliminary results<sup>3,4</sup> indicate that this technique is surprisingly accurate, at least for relatively small changes in optical thickness. This approximation involves first solving both the radiative transfer equation *and its adjoint* for a suitable base model atmosphere:

$$L_0 I_0 = Q, \quad (1a)$$

$$L_0^+ I_0^+ = R, \quad (1b)$$

where  $L_0$  is the base case transport operator,  $L_0^+$  its adjoint,<sup>5</sup> and  $I_0$  and  $I_0^+$  the corresponding forward and adjoint radiances,  $Q$  the source of radiation, and  $R$  the response function for the particular effect in question. Definitions are given in Refs. 1 and 2. The base case radiative effect,  $E$  may now be computed in two ways:<sup>1</sup>

$$E_0 = \langle R, I_0 \rangle \quad (2a)$$

or

$$E_0 = \langle I_0^+, Q \rangle, \quad (2b)$$

where angular brackets denote integration over all relevant space and angle coordinates (plus wavelength if necessary), which constitute the phase space of the problem.

If it is subsequently desired to determine the effect for a nearby atmosphere, described by a transport operator,  $L$ , then, to first order, Box et al<sup>2</sup> showed that

$$E \cong E_0 - \langle I_0^+, \Delta L I_0 \rangle, \quad (3)$$

where

$$\Delta L = L - L_0.$$

Equation (3) is reminiscent of first-order perturbation theory (Ref. 6, Chap. XVI), and it is natural to ask whether it is possible to improve the accuracy by going to higher-order terms. Unfortunately, standard perturbation expansions involve the summation over a complete set of eigenstates of the unperturbed (base case) operator. While considerable attention has been given to the spectrum of the transport operator in the neutron transport context,<sup>7</sup> this approach to transport problems has had little following amongst radiative transfer workers.<sup>8</sup>

For this reason, in this paper we have embarked on an alternate course, based on the Green's function approach.<sup>5</sup> Since the Green's function can effectively be described as the solutions of the transfer equation for all possible and appropriate sources, most existing radiative transfer codes should be capable of producing  $G$  with only moderate re-coding.

In the next section, we define the Green's function,  $G$ , and discuss some of its properties, including its interpretation as an inverse transport operator. This approach will lead us to Dyson's equation for  $G$  and hence to the full perturbation series (or Dyson series) for a radiative effect.

### THE GREEN'S FUNCTION

Green's function for a transport operator,  $L$ , may be defined as the solution of the transport problem corresponding to a delta function source in both position and direction.<sup>5</sup> We write

$$LG(\mathbf{r}_0, \boldsymbol{\Omega}_0 \rightarrow \mathbf{r}, \boldsymbol{\Omega}) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}_0). \quad (4a)$$

Suitable reduced Green's functions may be defined when the problem is restricted to be plane parallel (where we may replace  $\mathbf{r}, \mathbf{r}_0$  by  $z, z_0$ ), and azimuth independent (replace  $\boldsymbol{\Omega}, \boldsymbol{\Omega}_0$  by  $\mu, \mu_0$ ). The adjoint Green's function may be similarly defined by

$$L^+G^+(\mathbf{r}_0, \boldsymbol{\Omega}_0 \rightarrow \mathbf{r}, \boldsymbol{\Omega}) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}_0). \quad (4b)$$

For any source  $Q$ , the radiance is now given by

$$I = \langle G, Q \rangle. \quad (5)$$

Similarly, for any adjoint source (i.e., response function), the adjoint radiance,  $I^+$ , may be written [cf. Eq. (1b)] as

$$I^+ = \langle G^+, R \rangle. \quad (6)$$

We will often drop the angular brackets in these two expressions and regard  $G$  and  $G^+$  as *operators*, whereby phase space integration is implied.  $G$  and  $G^+$  are clearly related to the resolvent operators of  $L$  and  $L^+$  (see Refs. 6, 9). The effect in question, i.e., the one corresponding to the response function  $R$ , may now be determined via

$$E = \langle R, GQ \rangle = \langle G^+R, Q \rangle. \quad (7)$$

The equivalence of these two expressions<sup>1</sup> is guaranteed by the fundamental reciprocity relation<sup>5</sup>

$$G^+(\mathbf{r}_1, \boldsymbol{\Omega}_1 \rightarrow \mathbf{r}_2, \boldsymbol{\Omega}_2) = G(\mathbf{r}_2, \boldsymbol{\Omega}_2 \rightarrow \mathbf{r}_1, \boldsymbol{\Omega}_1). \quad (8)$$

$G$  and  $G^+$  both satisfy the reciprocity relation<sup>5</sup>

$$G(\mathbf{r}_1, \boldsymbol{\Omega}_1 \rightarrow \mathbf{r}_2, \boldsymbol{\Omega}_2) = G(\mathbf{r}_2, -\boldsymbol{\Omega}_2 \rightarrow \mathbf{r}_1, -\boldsymbol{\Omega}_1), \quad (9a)$$

$$G^+(\mathbf{r}_1, \boldsymbol{\Omega}_1 \rightarrow \mathbf{r}_2, \boldsymbol{\Omega}_2) = G^+(\mathbf{r}_2, -\boldsymbol{\Omega}_2 \rightarrow \mathbf{r}_1, -\boldsymbol{\Omega}_1). \quad (9b)$$

The most important property of Green's function which we now need is its interpretation as an inverse transport operator. Starting with the fundamental definition of Eq. (4), we operate with this product on an arbitrary source function  $F(\mathbf{r}_0, \boldsymbol{\Omega}_0)$  to obtain

$$\int d\mathbf{r}_0 \int d\boldsymbol{\Omega}_0 LG(\mathbf{r}_0, \boldsymbol{\Omega}_0 \rightarrow \mathbf{r}, \boldsymbol{\Omega})F(\mathbf{r}_0, \boldsymbol{\Omega}_0) = \int d\mathbf{r}_0 \int d\boldsymbol{\Omega}_0 \delta(\mathbf{r} - \mathbf{r}_0)\delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}_0)F(\mathbf{r}_0, \boldsymbol{\Omega}_0) = F(\mathbf{r}, \boldsymbol{\Omega}).$$

We now interchange the orders of  $L$  and integrations, remembering that  $L$  operates on the coordinates  $\mathbf{r}, \boldsymbol{\Omega}$ , to obtain

$$L\langle G, F \rangle = F(\mathbf{r}, \boldsymbol{\Omega}).$$

Since the source function,  $F$ , was arbitrary, what we have actually shown is that

$$LG = 1. \quad (10a)$$

That is to say,  $G$  is a right inverse of  $L$ . In an exactly analogous fashion, we may show that  $G^+$  is a right inverse of  $L^+$ .

We must now show that these Green's functions are each left inverses of their respective operators. Firstly, starting from Eq. (5) and operating on both sides with the transport operator  $L$ , we find (using operator notation)

$$I = GQ,$$

and therefore,

$$LI = LGQ = Q,$$

using Eq. (1) or Eq. (10). If we now operate on both sides of this equation with  $G$ , we obtain

$$GLI = GQ = I.$$

where we have used Eq. (5) again. Thus we have shown that

$$GL = 1. \tag{10b}$$

A simple alternative derivation may prove illuminating. Since  $G^+$  is a right inverse of  $L^+$ , we may write

$$L^+G^+ = 1.$$

If we now take the adjoint of this operator equation, we see that

$$(L^+G^+)^+ = 1,$$

i.e.

$$GL = 1.$$

### THE DYSON EQUATION FOR $G$

Let us assume now that we have been able to obtain the Green's function  $G_0$ , for some (perhaps artificial) base case transport operator  $L_0$ . As an extreme case,  $L_0$  may correspond to a vacuum atmosphere! We now wish to solve some alternate transport problem, characterized by the operator  $L$ . Clearly, we could achieve this aim if we could obtain the corresponding Green's function  $G$ . In fact, a relation between  $G$ ,  $G_0$  and  $\Delta L$  may be obtained as follows. With  $F$  an arbitrary source function, we consider

$$\begin{aligned} (G_0 - G\Delta LG_0)F &= G_0F - G(L - L_0)G_0F \\ &= G_0F - (GL)G_0F + G(L_0G_0)F \\ &= GF, \end{aligned}$$

i.e.

$$G = G_0 - G\Delta LG_0, \tag{11}$$

where we have twice used Eq. (10). Equation (11) is effectively Dyson's equation for  $G$ . The quantum mechanical version<sup>6,9</sup> differs in the sign, due to a sign difference in the definition of  $G$ .

Equation (11) is not a simple equation for  $G$  and is, in fact, an integral equation. Solution by direct means is almost certainly not practical, although it could be converted by a suitable quadrature to a very large matrix equation. Rather than attempting this task, we shall pursue the more normal approach to solving the Dyson equation<sup>9</sup> via iteration/perturbation, by successive substitution in the right side of Eq. (11):

$$\begin{aligned} G &= G_0 - G\Delta LG_0 \\ &= G_0 - G_0\Delta LG_0 + G\Delta LG_0\Delta LG_0 \\ &= G_0 - G_0\Delta LG_0 + G_0\Delta LG_0\Delta LG_0 \dots \end{aligned} \tag{12}$$

This equation represents a (hopefully convergent) power series in  $\Delta LG_0$ .

Before we consider the application of Eq. (12), we note that the preceding derivation can be used to produce the following results:

$$G = G_0 - G\Delta LG_0, \tag{11}$$

$$G = G_0 - G_0\Delta LG, \tag{13}$$

$$G^+ = G_0^+ - G^+\Delta L^+G_0^+, \tag{14}$$

$$G^+ = G_0^+ - G_0^+\Delta L^+G^+. \tag{15}$$

THE PERTURBATION EXPANSION FOR  $E$ 

Usually, what one is interested in obtaining is a certain radiative effect (or effects), such as a flux or a heating rate. In this case, we may insert Eq. (12) in Eq. (17) to obtain

$$\begin{aligned}
 E &= \langle R, GQ \rangle \\
 &= \langle R, G_0Q \rangle - \langle R, G_0\Delta LG_0Q \rangle + \langle R, G_0\Delta LG_0\Delta LG_0Q \rangle - \langle R, G_0\Delta LG_0\Delta LG_0\Delta LG_0Q \rangle + \cdots \\
 &= E_0 - \langle G_0^+R, \Delta LG_0Q \rangle + \langle G_0^+R, \Delta LG_0\Delta LG_0Q \rangle - \langle G_0^+R, \Delta LG_0\Delta LG_0\Delta LG_0Q \rangle + \cdots \\
 &= E_0 - \langle I_0^+, \Delta LI_0 \rangle + \langle I_0^+, \Delta LG_0\Delta LI_0 \rangle - \langle I_0^+, \Delta LG_0\Delta LG_0\Delta LI_0 \rangle + \cdots. \quad (16a)
 \end{aligned}$$

In this derivation, we have made use of the definition of an adjoint operator,<sup>1</sup> as well as Eqs. (5) and (6).

Equation (16a) can be seen to be the generalization of Eq. (3), which contains only the first two terms. Each higher-order term in Eq. (16a) differs from the preceding term by the inclusion of an extra  $\Delta LG_0$ , which involves two extra angular integrations and one spatial integration. By storing the function to the right of the comma, before performing the final phase-space integration with  $I_0^+$ , these additional integrations may be performed iteratively, so that it is not necessary to start each term from scratch.

As it stands, Eq. (16a) is best suited to situations where many effects are sought from only a few sources. This can be seen by the fact that  $I_0$ , which will clearly be different for each source (e.g., solar zenith angle), is the seed for the iteration procedure to compute the right-hand side of successive terms in the series. If it is desired to obtain only a small number of effects from a larger number of sources (e.g., surface flux for a range of zenith angles), then we may use instead

$$E = E_0 - \langle I_0, \Delta L^+ I_0^+ \rangle + \langle I_0, \Delta L^+ G_0^+ \Delta L_0^+ I_0^+ \rangle - \langle I_0, \Delta L^+ G_0^+ \Delta L^+ G_0^+ \Delta L^+ I_0^+ \rangle + \cdots. \quad (16b)$$

Equation (16b) may be obtained from Eq. (16a) simply by making use of the definition of an adjoint.

The number of terms required for the convergence of Eq. (16a) will naturally depend on the size of the perturbation  $\Delta L$ , and probably only future experience will be able to guide us in this area. Should the number of terms become excessive, it may become attractive to reconsider the matrix inversion option. However, it would be rare to attempt any form of perturbation calculation for only a single  $\Delta L$ : a direct computation for the required  $L$  would seem more logical.

It is more likely that perturbation would be employed when we are interested in a whole family of perturbations of the form  $\alpha \Delta L$ , where  $\alpha$  is a simple scalar multiplier which is expected to take on a range of values. Then, once the individual terms of Eq. (16a) were computed and stored, each could be multiplied by the appropriate power of  $\alpha$  for any perturbation in the family. If matrix inversion is employed, the entire process needs to be repeated for each value of  $\alpha$ . Nevertheless, matrix inversion for a representative  $\alpha$  may still be desirable as a cross check on convergence and computational economy.

## DISCUSSION

To date, Green's functions have received comparatively scant attention in the radiative transfer literature,<sup>10-13</sup> and much of that has been restricted to emergent radiation. It is certainly true that the computation of  $G(\mathbf{r}_0, \Omega_0 \rightarrow \mathbf{r}, \Omega)$  for a realistic  $L$  is a nearly impossible task, except for Monte Carlo simulation. However, it is not necessary to use this procedure in order to obtain a satisfactory  $G$ .

The simplest  $G_0$  is that for a vacuum  $L_0$ , where all emitted photons stream off to infinity. The case of a purely absorbing atmosphere is almost as easily treated (Ref. 14, p. 9) and certainly closer to reality. Preliminary results for first order perturbation theory<sup>3,4</sup>—Eq. (3)—indicate greatest accuracy when  $\Delta L$  involves no change in total cross-section, but only a change in phase function and/or single scattering albedo. Thus, selecting  $L_0$  to correspond to a purely absorbing atmosphere of appropriate optical thickness may result in reasonably rapid convergence of the perturbation series.

For the case of plane-parallel perturbations to a plane-parallel base atmosphere, both horizontal integrations are removable,  $G_0$  becomes a function only of  $z$ ,  $z_0$  and angles, and it is thus computable with most radiative transfer codes. The case of non-plane parallel perturbations to a plane-parallel base model also results in some simplifications. Both of these cases will be pursued elsewhere.

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#### REFERENCES

1. M. A. Box, S. A. W. Gerstl, and C. Simmer, "Application of the Adjoint Formulation to the Calculation of Atmospheric Radiative Effects," submitted to *Beitr. Phys. Atmos.* (1988).
2. M. A. Box, S. A. W. Gerstl, and C. Simmer, "Computation of Atmospheric Radiative Effects via Perturbation Theory," submitted to *Beitr. Phys. Atmos.* (1988).
3. M. A. Box, B. Croke, S. A. W. Gerstl, and C. Simmer, "Application of the Perturbation Theory for Atmospheric Radiative Effects I: Fluxes," submitted to *Beitr. Phys. Atmos.* (1988).
4. M. A. Box, B. Croke, S. A. W. Gerstl, and C. Simmer, "Application of the Perturbation Theory for Atmospheric Radiative Effects II: Heating Rates," submitted to *Beitr. Phys. Atmos.* (1988).
5. G. I. Bell and S. Glasstone, *Nuclear Reactor Theory*, §6.1, Van Nostrand-Reinhold, New York, NY (1970).
6. A. Messiah, *Quantum Mechanics*, pp. 712–715, North Holland, Amsterdam (1965).
7. K. M. Case and P. F. Zweifel, *Linear Transport Theory*, Addison-Wesley, Reading, MA (1967).
8. H. C. van de Hulst, *Astron. Astrophys.* **9**, 366 (1970).
9. J. M. Ziman, *Elements of Advanced Quantum Theory*, p. 124, Cambridge University Press, Cambridge (1969).
10. H. M. Domanus and A. C. Cogley, *JQSRT* **14**, 705 (1974).
11. A. C. Cogley, *JQSRT* **15**, 749 (1975).
12. S. Twomey, *JQSRT* **33**, 575 (1985).
13. H. Domke and E. G. Yanovitskij, *JQSRT* **36**, 175 (1986).
14. S. Chandrasekhar, *Radiative Transfer*, Dover, New York, NY (1960).