

# Retrieval of the albedo and phase function from exiting radiances with radiative perturbation theory

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We use radiative perturbation theory to develop a retrieval technique for determining the radiative properties of a scattering medium, such as the Earth's atmosphere, based on measurements of the radiation emerging at either the top or bottom of the medium. In a previous paper [J. Quant. Spectrosc. Radiat. Transfer **54**, 695 (1995)] we have shown the capacity of radiative perturbation theory to describe variations in exiting intensity as a linear combination of the parameters that characterize the scattering medium. Here we show that it is possible to set up a matrix relation such that the matrix inversion solves the inverse scattering problem. Using simulated data, we observe that the quality of the solution can be controlled by studying the singular values associated with the kernel matrix, obtaining in this way a stable solution, even in the presence of noise. © 1999 Optical Society of America

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## 1. Introduction

Monitoring the radiative properties of the atmosphere from ground-based or satellite measurements of radiation exiting at the boundaries is a complicated and computationally intensive problem. The required parameters appear directly in the radiative transfer equation; however, to retrieve them we must solve this equation inversely with respect to these parameters. To solve this inverse scattering problem, several methods have been described in the literature. Research by Kamiuto<sup>1,2</sup> referred to as the emerging-intensity fitting method obtained information about the single-scattering albedo and the asymmetry factor. However, because this method assumed that the phase function follows the Henyey-Greenstein<sup>3</sup> model, it was not possible to retrieve the full details of the phase function. An important improvement has been achieved by Kamiuto when a new inversion method was proposed to estimate a more accurate phase function.<sup>4</sup> However, that research involves a large number of iterations that is time-consuming, and the results also deteriorate in the presence of noise.

Independently of the author and technique used,

the solution of the inverse scattering problem involves solving two separate problems simultaneously. First we have to obtain a suitable mathematical description of the physical problem: This is the so-called direct problem. Then we have to tailor an inversion method to select the physically allowable solution from the infinite set of mathematically correct solutions obtained from the inversion of the equations that describe the direct problem.

In this paper we develop an inversion technique based on radiative perturbation theory.<sup>5-7</sup> In a previous paper<sup>8</sup> we showed that, for a vertically homogeneous atmosphere without ground reflectances and fixed total optical thickness, radiative perturbation theory was able to predict the differences in the exiting radiances between different atmospheric optical models, based on a detailed calculation for one of the models (referred to as the base model), plus a knowledge of the difference between the two model transport operators (see below). We now intend to apply these ideas in the following way. We make an initial guess as to the optical model of the atmosphere (scattering albedo, phase function). From this we compute the expected exiting radiances, which, in most cases, differ from the actual measurements. We then work backward, using perturbation theory, by inverting the differences between observed and predicted intensities to obtain information on the difference between the true and the assumed optical models. Note that no assumptions need be made about the shape of the particles.

From the mathematical point of view, the direct

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problem is described by a set of  $K$  linear equations (one for each observational direction) in  $L$  unknowns (corresponding to  $L-1$  different Legendre order phase function expansion coefficients, plus the single-scattering albedo). We represent this system of linear equations by a matrix of dimension  $L \times K$ , called the kernel matrix. Inverting this matrix allows us to solve the inverse problem.

One of the most important problems we must address is the selection of the appropriate dimensions of the kernel matrix. The selection of the  $K$  dimension involves not just the selection of the number of measurements, but also the determination of those observational directions that contain more information. We address the study of this problem in our next paper, assuming in this research that we have a set of uniformly distributed data in all directions and that the entire data set will be considered without distinction for all the examples discussed below.

The determination of the appropriate  $L$  dimension of the kernel matrix, and hence the magnitude of the elements of the kernel matrix, allows us to control the stability of the solution.<sup>9</sup> Simultaneously, the inclusion of a damping parameter in the calculation of the inverse matrix using a least-squares method will improve the accuracy in the calculation of the matrix inverse. Thus the maximum Legendre order in the phase function expansion and the damping parameter are elements that determine the mathematical structure of the matrix itself that can be tested using, for example, singular-values decomposition.<sup>10</sup>

In Section 2 we use radiative perturbation theory to develop the necessary equations to solve the inverse scattering problem, outlining at the same time the numerical problems involved in the inversion process. In Section 3 we use synthetic data to analyze the accuracy of the retrieved model parameters obtained for different  $L$  dimensions of the kernel matrix and different values of the damping parameter. We discuss the correlation between the goodness of the different solutions and the set of singular values associated with the kernel matrix used for each inversion. We also study the effect of added noise and the variation in the total optical thickness on the quality of the retrieved coefficients. Finally, we demonstrate the value of selecting the right base case to obtain good results after successive inversions.

We emphasize at this point that our entire philosophy is different from the standard aureole inversion problem<sup>11</sup> in which one usually attempts to extract the aerosol size distribution<sup>12</sup> and occasionally also the refractive index or single-scattering albedo.<sup>13</sup> In particular, we make no assumptions as to the shape of the scatterers, and we do not use Mie theory at any stage. Our interest is focused precisely on the key optical quantities that are needed to determine the radiative or climatic effect of the aerosol particles<sup>14</sup> and not on their underlying physical properties.

Note that in the present research we address only a restricted problem in which the total optical thickness is assumed known, the medium is assumed uniform, and ground reflection is ignored. This is not

an unreasonable set of assumptions for ground-based measurements at near-infrared wavelengths over a dark surface. These restrictions will be progressively removed in future papers.

## 2. Formulation of the Inverse Problem

### A. Radiative Perturbation Theory

The radiative transfer equation that describes the interaction between solar radiation and the atmospheric components can be expressed<sup>5</sup> in the following form:

$$LI = Q, \quad (1)$$

where  $Q$  is the source (in this case the extraterrestrial solar flux),  $I$  is the resultant radiance distribution, and  $L$  is the transport operator defined by

$$L = \mu \frac{\partial}{\partial z} + \sigma_t(z) - \sigma_s(z) \int d\Omega' p(z, \Omega' \rightarrow \Omega) \circ \quad (2)$$

The component characteristics of the medium are incorporated in the profiles of extinction  $\sigma_t(z)$ , scattering  $\sigma_s(z)$ , and the phase function  $p(z, \Omega' \rightarrow \Omega)$ , which is assumed to be expandable in a series of Legendre polynomials  $P_l(\theta)$  as

$$p(\theta) = \sum_{l=0}^L \chi_l P_l(\theta) / 4\pi. \quad (3)$$

If we consider measurements of the exiting intensity at the bottom of the atmosphere, then it may be assumed that we also know the total optical thickness, and therefore the perturbation of the intensity is caused by the perturbation of the scattering characteristics  $\sigma_s(z)$  and  $\chi_l$ , whereas the total optical thickness remains constant. In the case of measurements at the top of the atmosphere, this may or may not be true.

In our previous paper<sup>8</sup> we demonstrated that radiative perturbation theory is able to account for changes or perturbations in the exiting radiance  $\Delta E \equiv E^{pc} - E^{bc}$  caused by perturbations in the phase function expansion coefficients  $\Delta\chi_l = \chi_l^{pc} - \chi_l^{bc}$  and in the scattering cross section  $\Delta\sigma_s(z) = \sigma_s^{pc}(z) - \sigma_s^{bc}(z)$ . Superscripts pc and bc refer to the perturbed and base case (or model), respectively.

If we restrict ourselves to the simplest case of a uniform slab medium, with a uniform perturbation at all levels, then  $\Delta E$  can be expressed in the following form<sup>8</sup>:

$$\Delta E(\Omega_{\text{obs}}, \Omega_{\text{solar}}) = \pi F_0 \sum_m \cos m\varphi_0 \sum_{l \geq m} \Delta(\omega_0 \chi_l) Z_{lm}, \quad (4)$$

where  $\Omega_{\text{obs}} = (\theta_{\text{obs}}, \varphi_{\text{obs}})$  and  $\Omega_{\text{solar}} = (\theta_{\text{solar}}, \varphi_{\text{solar}})$  are the angular positions of the sensor and the Sun, respectively, and  $\varphi_0$  denotes the difference between the solar and the observational azimuth angles.  $Z_{lm}$  is a

tridiagonal array ( $m \leq l$ ) that depends only on the base model. We can reexpress Eq. (4) as

$$\begin{aligned} \Delta E(\Omega_{\text{obs}}, \Omega_{\text{solar}}) &= \sum_l \left( \sum_m \pi F_0 \cos m\varphi_0 Z_{lm} \right) \Delta(\omega_0 \chi_l) \\ &= \sum_l A_l(\Omega_{\text{obs}}, \Omega_{\text{solar}}) \Delta \eta_l, \end{aligned} \quad (5)$$

where  $A_l = \sum_m \pi F_0 \cos m\varphi_0 Z_{lm}$ ,  $\eta_l = \omega_0 \chi_l$ , and  $\Delta \eta_l = \eta_l^{\text{pc}} - \eta_l^{\text{bc}}$  is the perturbation in the  $l$ th coefficient  $\eta_l$ . The coefficients  $A_l$  are independent of any atmospheric model perturbation and are computed from the assumed first guess model or base model.

The solution of the direct problem is given by the linear relation shown in Eq. (5): We may know the variation of the exiting signal as a function of the variation of the system transfer equation. The inverse problem we are interested in is the so-called inverse scattering problem: We want to determine the radiative properties of the scattering medium, specifically the single-scattering albedo and phase function.

### B. Inverse Problem

We are interested in inverting Eq. (5) with respect to  $\Delta \eta_l$ , thus obtaining an expression for these coefficients as a function of the measured intensities. We give the measured intensity the role of perturbed intensity. This intensity is associated with the unknown profile that we want to retrieve, whereas the base case intensity is computed from an assumed first guess profile. In a real application we have a set of measurements in which the observational coordinates change from one datum to another. However, for all of them the Sun position is almost the same. Therefore we can assume that the Sun position is fixed while the measurements are performed at the top or bottom of the atmosphere. Our goal is to obtain a mathematical expression for the atmospheric model parameters as a function of the measured intensities for different observational angles. This is the inverse problem.

To achieve this goal we must assume a copy of Eq. (5), corresponding to each of the  $K$  different observational directions, and hence we establish a system of  $K$  linear equations, each equation with  $L$  terms where  $L - 1$  is the maximum order used in the Legendre expansion of the phase function.

We can express this set of equations in matrix form as

$$\Delta \mathbf{E} = \mathbf{A} \Delta \boldsymbol{\eta}, \quad (6)$$

where  $\mathbf{A}$  is the kernel matrix of dimension ( $K \times L$ ) with elements  $A_{lk} \equiv A_l(\Omega_k)$ , which relate the change in the intensity observed in direction  $\Omega_k$  and a perturbation suffered by the  $l$ th coefficient  $\eta_l$ .

$\Delta \mathbf{E}$  is now a vector of dimension  $K$ , where each component is the difference between the observed datum ( $E^{\text{pc}}$ ) and the value computed from our initial guess ( $E^{\text{bc}}$ ).  $\Delta \boldsymbol{\eta}$  is a vector of dimension  $L$  that describes the difference in the scattering properties be-

tween the Legendre expansion coefficients of the new (unknown) model and the initial guess model.

By inverting Eq. (6) we are able to obtain information about the unknown profile  $\eta^{\text{pc}}$  from a base model  $\eta^{\text{bc}}$  and from the difference between the exiting intensities as shown here:

$$\boldsymbol{\eta}^{\text{retr}} = \mathbf{B} \Delta \mathbf{E} + \boldsymbol{\eta}^{\text{bc}}, \quad (7)$$

where  $\mathbf{B}$  is the inverse of  $\mathbf{A}$ . Therefore to obtain the unknown profile we only need to know the intensity  $E^{\text{pc}}$ , measured at the top or bottom of the atmosphere for  $K$  different observational directions. The selection of the base case is completely arbitrary, although it is clearly desirable that it should not represent too large a deviation. This is a necessary condition for the first-order approximation to be valid because we have assumed a perturbation approach.

Using the base phase function in the radiative transfer program, we compute the exiting intensity  $\mathbf{E}^{\text{bc}}$  for the same Sun and observation coordinates that correspond to the real intensity. Thus, knowing  $\eta^{\text{bc}}$ , we are able to retrieve the unknown coefficients  $\boldsymbol{\eta}^{\text{retr}}$  and the single-scattering albedo as

$$\Delta \eta_0^{\text{retr}} = \Delta \omega_0^{\text{retr}} = \omega_0^{\text{retr}} - \omega_0^{\text{bc}}, \quad (8)$$

where  $\Delta \eta_0^{\text{retr}}$  is the first retrieved coefficient. The retrieved albedo is calculated as

$$\omega_0^{\text{retr}} = \Delta \omega_0^{\text{retr}} + \omega_0^{\text{bc}}, \quad (9)$$

and the retrieved scattering expansion coefficients are calculated as

$$\chi_l^{\text{retr}} = (\Delta \eta_l^{\text{retr}} + \eta_l^{\text{bc}}) / \omega_0^{\text{retr}}. \quad (10)$$

### C. Matrix Inversion

We noted in Subsection 2.B that the dimension and structure of the kernel matrix are determined by the maximum order  $L$  that has been chosen for the phase function representation and the selection of the set of  $K$  observational directions. This matrix can be seen as the interrelation or mapping between the  $L$ -dimensional space associated with the model and the  $K$ -dimensional space associated with the measurements. The success of the inversion will depend on the capability of the kernel matrix to describe the model space from the measurement space<sup>10</sup> and on the accuracy of the calculation of  $\mathbf{B}$ .

Because  $\mathbf{A}$  is, in principle, nonsquare, the inverse of the kernel matrix can be obtained from the well-known least-squares inverse method:

$$\mathbf{B} = (\mathbf{A}^t \mathbf{A})^{-1} \mathbf{A}^t, \quad (11)$$

where  $\mathbf{B}$  is called the pseudoinverse or generalized inverse of the matrix  $\mathbf{A}$ . The matrix  $\mathbf{A}^t \mathbf{A}$  can sometimes be singular or nearly singular, so the inverse of this matrix does not exist or is inaccurately computed.<sup>9,10</sup> The remedy to overcome such a situation has been proposed in the damped least-squares inversion

method; in this case the inverse matrix is calculated as

$$\mathbf{B} = (\mathbf{A}^t \mathbf{A} + \gamma \mathbf{I})^{-1} \mathbf{A}^t, \quad (12)$$

where  $\gamma$  is the damping parameter and  $\mathbf{I}$  is the identity matrix. The arbitrary factor  $\gamma$ , added to the main diagonal of the matrix  $\mathbf{A}^t \mathbf{A}$ , increases the magnitude of the eigenvalues of the matrix. This prevents the inversion from blowing up, so the solution remains stable.

The challenge now is to select values for both the kernel matrix dimension and  $\gamma$  to obtain accurate solutions. It is well known<sup>9,10</sup> that the numerical errors produced in the calculation of the inverse of a matrix are due to the presence of small singular values associated with the matrix. Therefore a criterion to select  $K$ ,  $L$ , and  $\gamma$  can be based on finding the  $(K, L, \gamma)$  combination that produces the largest singular values of the kernel matrix.

### 3. Inversion of Synthetic Data

The measured intensity was simulated using the Lenoble Haze-L phase function expansion coefficients<sup>15</sup> and a scattering albedo of  $\omega_0^{\text{pc}} = 0.9$ . The first guess and the kernel matrix were computed using the Henyey–Greenstein model<sup>3</sup> (just one parameter,  $g$ , is necessary to characterize this phase function).  $g$  was set to 0.75 and the scattering albedo  $\omega_0^{\text{bc}}$  was set to 1.0. Three different optical thicknesses were considered: 0.2, 0.5, and 1.0. (Remember that this quantity is assumed known.)

We start with the case of observations at the bottom of the atmosphere, as under this situation we are usually assured that the total optical thickness is known. The simulated data were calculated for four different observational zenith angles ( $4^\circ$ ,  $32^\circ$ ,  $59^\circ$ ,  $87^\circ$ ) in seven different azimuth directions ( $0^\circ$ ,  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$ ,  $120^\circ$ ,  $150^\circ$ ,  $180^\circ$ ). The Sun position was fixed for all calculations at  $59^\circ$  from the zenith and in the azimuth plane. The complete set of 28 observational angles was used for the analysis of the results. This set of angles is not necessarily optimum nor do we suggest that a set this large is necessarily required. The four zenith angles were chosen as (approximately spaced) quadrature directions in our computer code, whereas the azimuth angles were chosen to be equally spaced. (Observations on the other side of the solar plane would obviously provide no additional information.) The actual set of angles to be selected is likely to be partially or totally dictated by the technology involved, especially in the case of satellite observation. In those situations (especially ground based) in which one has more flexibility to choose observation directions after the data are taken, it would obviously be appropriate to ignore multiple measurements that differed little and concentrate on those measurements in which the radiance was varying most rapidly. (We address the question of information content, at least in part, in a companion paper.<sup>16</sup>)

**Table 1. Percentage Error in the Retrieved Scattering Coefficients for Different  $\gamma$  Values and  $L = 7$**

Order	$\gamma = 0$	$\gamma = 0.001$	$\gamma = 0.01$	$\gamma = 0.1$	$\gamma = 1.0$
$\omega_0$	0.64	1.11	1.40	1.42	1.99
$\chi_1$	1.39	2.4	1.16	0.81	0.07
$\chi_2$	4.47	3.13	2.26	3.05	4.94
$\chi_3$	3.68	3.38	2.16	2.98	4.38
$\chi_4$	10.95	3.00	4.55	3.90	4.05
$\chi_5$	5.53	2.87	4.06	2.35	1.86
$\chi_6$	15.8	2.21	6.02	0.59	0.72

#### A. Kernel Matrix Analysis

Before attempting to perform the actual inversion, it is important to study and set the characteristics of the kernel matrix to control the numerical errors involved in the calculation. We present a criterion to select the  $L$  dimension and the damping parameter  $\gamma$  to introduce the smallest error in the calculation of the matrix  $\mathbf{B}$ . To illustrate this selection criterion we analyze the inversion results for a case in which the total optical thickness equals 1.0. (This is not a constraint; any other case could have been used.)

We calculated the matrix  $\mathbf{B}$  using different values of  $\gamma$  for a fixed value of the  $L$  dimension. We verified that the retrieved coefficients obtained using  $\gamma = 0.1$  are more accurate, on average, than those obtained for any other value of  $\gamma$  (see Table 1). This behavior was found to be independent of the considered  $L$  dimension.

We can correlate this result with the normalized singular values associated with the inverse of the kernel matrix. In Table 2 we can observe, for  $L = 7$ , the different set of singular values of the matrix  $\mathbf{B}$  computed for different  $\gamma$  values. It is clear that for  $\gamma = 0.1$  the singular values are generally greater than the values for any other  $\gamma$ , justifying the low percentage errors found in the solution, as shown in Table 2.

Once the  $\gamma$  value was selected, we have to decide which is the appropriate  $L$  dimension of the kernel matrix to be used in the inversion. If the  $L$  value is small then we obtain a poor representation of the phase function; if it is too large then it produces small singular values, affecting the calculation of  $\mathbf{B}$ . Table 3 shows the different sets of singular values obtained with  $\gamma = 0.1$  for several  $L$  dimensions. We can observe that the additional singular values of

**Table 2. Normalized Singular Values of the  $B$  Matrix Calculated for  $L = 7$  and Different  $\gamma$  Values**

Order	$\gamma = 0$	$\gamma = 0.001$	$\gamma = 0.01$	$\gamma = 0.1$	$\gamma = 1.0$
1	1.0000	1.000	1.0000	1.0000	1.0000
2	0.2272	0.2515	0.8783	0.9658	0.7600
3	0.1645	0.1826	0.7276	0.8899	0.5781
4	0.1216	0.1351	0.5657	0.7397	0.4622
5	0.0647	0.0720	0.4311	0.6303	0.3505
6	0.0319	0.0355	0.2203	0.2053	0.2577
7	0.0094	0.0104	0.0656	0.1925	0.0595

**Table 3. Normalized Singular Values of the Kernel Matrix Calculated for  $\gamma = 0.1$  and Different  $L$  Dimensions**

Order	$L = 7$	$L = 14$	$L = 19$	$L = 20$	$L = 22$
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.9658	0.9360	0.9294	0.9290	0.9307
3	0.8899	0.9297	0.9240	0.9220	0.9187
4	0.7297	0.7396	0.7399	0.7394	0.7448
5	0.6303	0.6284	0.6434	0.6430	0.6636
6	0.2053	0.6137	0.6086	0.6078	0.6068
7	0.1925	0.5277	0.5358	0.53645	0.5373
8		0.4143	0.4859	0.4864	0.4921
9		0.3098	0.3440	0.3746	0.3877
10		0.2887	0.3211	0.3217	0.3533
11		0.2471	0.3100	0.3198	0.3300
12		0.2003	0.2701	0.2733	0.2742
13		0.0283	0.2340	0.2479	0.2528
14		0.0130	0.1992	0.1989	0.1987
15			0.1578	0.1605	0.1787
16			0.0383	0.1122	0.1435
17			0.0175	0.0176	0.0964
18			0.0024	0.0098	0.0187
19			0.0007	0.0017	0.0131
20				0.0004	0.0028
21					0.0011
22					0.0000

higher order are close to 0 for  $L$  dimensions above approximately 20.

The quality of the corresponding solutions is shown in Table 4, where we tabulated the percentage error of the retrieved coefficients for the different  $L$  dimensions of the kernel matrix. Reading Table 4 we can easily recognize the errors introduced by small sin-

**Table 4. Percentage Error in the Expansion Coefficients and Scattering Albedo Retrieved from  $\gamma = 0.1$  and Different  $L$  Dimensions of the Kernel Matrix<sup>a</sup>**

Order	$L = 7$	$L = 14$	$L = 19$	$L = 20$	$L = 22$
$\omega_0$	1.42	1.44	1.44	1.45	1.45
1	0.81	0.78	0.76	0.76	0.75
2	3.05	2.98	2.94	2.94	2.92
3	2.98	2.87	2.82	2.82	2.79
4	3.90	3.70	3.65	3.65	3.60
5	2.35	2.08	2.02	2.01	1.97
6	0.59	0.21	0.15	0.14	0.09
7		1.81	1.86	1.88	1.95
8		4.69	4.74	4.76	4.87
9		6.43	6.49	6.51	6.62
10		8.76	8.83	8.87	8.95
11		8.50	8.59	8.61	8.69
12		9.37	9.46	9.49	9.59
13		7.97	8.09	8.13	8.25
14			7.67	7.72	7.85
15			6.10	6.14	6.28
16			3.41	3.44	3.58
17			0.77	0.71	0.55
18			6.87	6.70	6.42
19				12.01	11.55
20					19.81
21					25.05

<sup>a</sup>Measured intensities at the bottom of the atmosphere for  $\tau_{\text{total}} = 1.0$ .

**Table 5. Percentage Error in the Expansion Coefficients and Scattering Albedo Retrieved from Intensities at the Bottom of the Atmosphere for Different Optical Thicknesses and  $\gamma = 0.1$**

Order	$\tau_{\text{total}} = 1.0$	$\tau_{\text{total}} = 0.5$	$\tau_{\text{total}} = 0.2$
$\omega_0$	1.35	1.11	1.44
1	0.36	0.40	0.76
2	3.00	2.58	2.94
3	2.99	2.66	2.82
4	3.29	3.30	3.65
5	1.49	1.72	2.02
6	0.20	0.37	0.15
7	1.93	1.60	1.86
8	4.20	4.11	4.74
9	5.53	5.61	6.49
10	8.03	8.09	8.83
11	6.99	7.49	8.59
12	8.77	8.76	9.46
13	9.23	7.78	8.09
14		7.97	7.67
15		6.07	6.10
16		1.72	3.41
17		3.59	0.77
18			6.87

gular values for  $L$  greater than 19. Therefore we can specify that if we are seeking solutions with errors smaller than 10%, we have to select those kernel matrices that present singular values greater than 0.01. In this specific example,  $L$  has to be smaller than 19.

#### B. Retrieval for Different Total Optical Thicknesses

Following the above-described criterion to select the matrix  $\mathbf{B}$  and consequently the solution, we analyze in this subsection the influence of the total optical thickness on the retrieval of the scattering characteristics. Table 5 shows the accuracy of the retrieved coefficients for different assumed optical thicknesses. Analyzing Table 5, we found that the number of coefficients we can retrieve with an error of less than 10% is reduced when we increase the total optical thickness. In this example we are able to retrieve 13, 17, and 19 coefficients for  $\tau_{\text{total}} = 1.0, 0.5,$  and  $0.2,$  respectively. In each case the smallest singular value was just greater than 0.01.

We also found that the retrieved coefficients are not sensitive to noise that is added to the signal. In Table 6 we present results of adding 5% noise to the measured intensity. As can be seen, the effect on the retrieval is negligible.

#### C. Intensities at the Top of the Atmosphere

If we have access to some extra information not derived from the original set of measurements—for example, direct measurement of total optical thickness—then it is possible to retrieve scattering coefficients and albedo from intensities measured at the top of the atmosphere. Table 7 shows the retrieved coefficients for different total optical thicknesses derived from intensities measured in the same set of directions as described above, but now with the

**Table 6. Retrieved Coefficients from Intensities at the bottom of the Atmosphere with 0% and 5% of Added Noise<sup>a</sup>**

Order	True Value	1st Guess	0% Noise	5% Noise
$\omega_0$	0.900	1.000	0.913	0.913
1	2.412	2.250	2.394	2.402
2	3.230	2.812	3.135	3.121
3	3.372	2.953	3.277	3.282
4	3.230	2.847	3.112	3.113
5	2.892	2.610	2.834	2.823
6	2.494	2.313	2.490	2.481
7	2.112	2.002	2.151	2.135
8	1.746	1.701	1.829	1.823
9	1.444	1.426	1.538	1.537
10	1.174	1.182	1.278	1.267
11	0.963	0.971	1.046	1.038
12	0.779	0.791	0.853	0.851
13	0.638	0.641	0.689	0.685
14	0.516	0.516	0.556	0.552
15	0.422	0.414	0.447	0.448
16	0.343	0.330	0.355	0.354
17	0.279	0.263	0.277	0.272
18	0.229	0.208	0.213	0.205

<sup>a</sup> $\tau_{\text{total}} = 1.0$ .

sensor located at the top of the atmosphere. The criterion for the selection of  $\gamma$  and  $L$  dimension of the kernel matrix was the same as that described in Subsection 3.A. The difference with respect to the measurements at the bottom is the threshold allowed for the singular values. It was found that, in the case of measurements at the top of the atmosphere, to obtain solutions with percentage errors lower than 10%, we have to restrict the magnitude of the singular values to be larger than 0.05. Again this result was found to be independent of the total optical thickness considered; however, in this case the number of retrieved coefficients was the same for all total optical thicknesses, namely 5. The corresponding solutions are shown in Table 7.

We can observe that, on average, the percentage error of the first few retrieved coefficients increases with the value of the total optical thickness. This effect is an indication of the sensitivity of the intensity measured at the top of the atmosphere to the scattering details.

**Table 7. Expansion Coefficients and Scattering Albedo Retrieved from Intensities at the Top of the Atmosphere for Different Optical Thicknesses**

Order	True Value	1st Guess	$\tau_{\text{total}}$		
			$\tau_{\text{total}} = 0.2$	$\tau_{\text{total}} = 0.5$	$\tau_{\text{total}} = 1.0$
$\omega_0$	0.900	1.000	0.907	0.907	0.910
1	2.412	2.250	2.395	2.389	2.378
2	3.230	2.812	3.145	3.156	3.177
3	3.372	2.953	3.278	3.283	3.331
4	3.230	2.847	3.156	3.159	3.207
5	2.892	2.610	2.877	2.873	2.610

**Table 8. Legendre Expansion Coefficients and Albedo Retrieved after Successive Iterations from Intensities at the Top of the Atmosphere<sup>a</sup>**

Order	True	1st Guess	1st Inversion	2nd Inversion	3rd Inversion
$\omega_0$	0.900	1.000	0.910	0.900	0.899
1	2.412	2.250	2.378	2.412	2.411
2	3.230	2.812	3.177	3.232	3.228
3	3.372	2.953	3.331	3.375	3.369
4	3.230	2.847	3.207	3.237	3.229
5	2.892	2.610		2.904	2.892
6	2.494	2.313		2.251	2.500
7	2.112	2.002		2.132	2.119
8	1.746	1.701		1.774	1.760
9	1.444	1.426		1.468	1.455

<sup>a</sup> $\tau_{\text{total}} = 1.0$ .

#### D. Retrieval after Successive Iterations

The base case or first guess coefficients are essentially arbitrary, but this arbitrariness can be reduced to some extent. We computed a family of intensities using the Henyey–Greenstein phase function expansion coefficients with different asymmetry parameter values. The minimum difference between computed and measured intensities was obtained for  $g = 0.75$ ; the corresponding Henyey–Greenstein phase function expansion coefficients for this case were then selected to be used as the first guess.

This first almost arbitrary approach can be improved if we do a second inversion using the coefficients retrieved after the first inversion as a new base case. However, this new base case will be a hybrid profile with a certain number of new coefficients that have been recently corrected by the inversion process, whereas the others belonging to the assumed first guess profile will remain unchanged. The inversion of the kernel matrix calculated from this hybrid profile will produce better results than the one obtained after the first inversion. This happens because we are now using a base case that is not as arbitrary as the first guess. To show the process of convergence after successive iterations we chose the case in which  $\tau_{\text{total}} = 1.0$  and for the top of the atmosphere. As indicated by the results we have presented so far, this is the case that produces the worst retrieved coefficients for the first inversion. Table 8 shows the improvement of the retrieved coefficients' accuracy for measurements at the top of the atmosphere, as well as the increment of the number of retrieved coefficients after each successive iteration.

Computing the exiting intensity for each successive hybrid profile, which from now on we call the retrieved intensity, we can evaluate the convergence of the iteration process by observing how the retrieved intensity after each iteration becomes closer and closer to the measured intensity. For example, Table 9 shows the differences between retrieved and measured intensities after each inversion, obtained from intensities at the top of the atmosphere. We note that these differences are smaller when we increment the number of iterations; we can consider

**Table 9. Differences between the Measured and Retrieved Intensities at the Top of the Atmosphere after Each Iteration<sup>a</sup>**

$(\theta_{\text{obs}}, \varphi_{\text{obs}})$	$E - E^{\text{bc}}$	$E - E^{\text{retr1}}$	$E - E^{\text{retr2}}$	$E - E^{\text{retr3}}$
$(87^\circ, 180^\circ)$	-0.0194	-0.00675	-0.000731	-0.0000710
$(59^\circ, 180^\circ)$	-0.0184	-0.0294	-0.00124	-0.0000681
$(31^\circ, 180^\circ)$	-0.0166	-0.00243	-0.000260	-0.0000127
$(4^\circ, 180^\circ)$	-0.0258	-0.00144	-0.000303	-0.00000378
$(4^\circ, 0^\circ)$	-0.0296	-0.00604	-0.000126	-0.0000155
$(31^\circ, 0^\circ)$	-0.0500	-0.00767	-0.000381	-0.0000643
$(59^\circ, 0^\circ)$	-0.0930	-0.00338	-0.000169	-0.0000626
$(87^\circ, 0^\circ)$	-0.898	-0.0560	-0.00163	-0.0000365

<sup>a</sup> $\tau_{\text{total}} = 1.0$ .

this fact a good indicator of the capabilities of the inversion procedure.

#### 4. Discussion

The results we have presented show that radiative perturbation theory offers a suitable platform to solve the problem of retrieving the single-scattering albedo and the phase function details. Remember that we have not assumed Mie theory, and hence we have not assumed spherical particles. We retrieved the single-scattering albedo and the expansion coefficients of the phase function, which are what we need for radiative transfer calculations (e.g., aerosol direct forcing<sup>14</sup>).

We have shown that it is possible to establish a system of equations represented by a kernel matrix, the inversion of which leads to a stable solution. The selection of the proper conditions ( $\gamma$  and  $L$  dimension) for the inversion, and consequent control of the accuracy of the solution, can be done by analyzing the set of singular values associated with the kernel matrix.

Multidirectional measurements of exiting radiances are now available from the Cimel instrument<sup>17</sup> that is now in widespread use. (This instrument also provides the optical thickness, which is required for our technique.) A number of algorithms have already been proposed to extract the maximum amount of information from such a data set, although all that we have seen to date either assume spherical particles or require iterative solutions of the radiative transfer equation. We believe that our procedure is superior for this reason. (Note also that our technique returns the aerosol optical properties at each wavelength for which such measurements are available.)

Space-based multidirectional measurements of exiting radiances were available for nearly one year from the polarization and directionality of the Earth's reflectances (POLDER) instrument<sup>18</sup> and will again be available within a few years from a combination of both the POLDER2 and the Earth Observing System moderate-resolution imaging spectrometer and the multiangle imaging spectroradiometer.<sup>19</sup> However, these instruments do not (directly) provide the aerosol optical depth and are likely to suffer from the uncertainty in the bidirectional reflectance distribu-

tion function of the underlying surface (at least over land). In the case of POLDER, the data set includes the multidirectional polarization at three wavelengths: We believe that an inversion technique based on perturbation theory for polarization is likely to be the most powerful approach possible, and we plan to address this task shortly.

It may be challenging for the reader to find that such a large number of coefficients can be retrieved after just one inversion. We believe that the radiative perturbation approach presented here is a powerful tool to be considered in the solution of the inverse scattering problem. Support for this conclusion can be found by looking at the low sensitivity to the added noise that was shown in Table 6. In addition, the low computational time involved in this inversion method allows further iterations, improving the accuracy of the results because of the fact that the process apparently converges to the perturbed profile.

To improve the inversion method, a diversity of new questions need to be answered. For example, in this paper a set of uniformly distributed data was selected, but because the Sun position determines the position of the pattern of the scattering phase function, we expect each of the data points to contain different information, requiring us to address the problem of selecting the best angular data. Many authors<sup>9,10</sup> have developed important tools to analyze how many pieces of information are available in a given set of data. Using this information we could estimate *a priori* the most suitable  $K$  dimension(s) for the kernel matrix. In future research we plan to investigate the spatial data selection and the number of necessary pieces of information needed to retrieve the coefficients in an efficient way. In addition we will extend the use of perturbation theory to include the total optical thickness and surface albedo as unknowns.

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