Asymptotic behavior of the Mie-scattering amplitude

Phil Attard, Michael A. Box, and Gary Bryant

School of Physics, University of New South Wales, Kensington, New South Wales 2033, Australia

Bruce H. J. McKellar

School of Physics, University of Melbourne Parkville, Victoria, 3052, Australia

Received January 18, 1985; accepted September 11, 1985

The asymptotic form of the Mie-scattering amplitude is of interest for a number of reasons, perhaps the most important of which is to reduce significantly the computational effort involved when the size parameter becomes large. Recently, Nussenzveig and Wiscombe [Phys. Rev. Lett. **45,** 1490 (1980)] published an asymptotic expansion for the Mie extinction efficiency based on complex-angular-momentum theory. In this paper, we show how this result may be generalized to the full forward-scattering amplitude.

In a recent paper, Nussenzveig and Wiscombe[1] presented asymptotic approximations to the Mie efficiency factors for extinction, absorption, and radiation pressure derived from complex-angular-momentum theory (otherwise known as the Watson transform). In particular, their approximation for the extinction efficiency may be written as

$$Q_{\text{ext}} = 2 + 1.9923861x^{-2/3} - \frac{8}{x}\,\text{Im}\left\{\frac{m^2+1}{4(m^2-1)^{1/2}} - F\exp[-2i(m-1)x] - \sum_{j=1}^{\infty}\left(\frac{m-1}{2j+1-m}\right)\left(\frac{m-1}{m+1}\right)^{2j}\exp[-2i(m-1+2jm)x]\right\} - 0.7153537x^{-4/3} + 0.3320643\,\text{Im}\left[e^{-i\pi/3}\frac{m^2+1}{(m^2-1)^{3/2}}\times(2m^4-6m^2+3)\right]x^{-5/3} + 0(x^{-2}) + \text{ripple}, \tag{1}$$

where

$$F = \frac{m^2}{(m+1)(m^2-1)}\left[1 - \frac{i}{2x}\left(\frac{1}{m-1} - \frac{m-1}{m}\right)\right],$$

$x = kr = 2\pi r/\lambda$ is the prize parameter, and $m = n - i\kappa$ is the complex refractive index. [Note that our sign convention is the conjugate of that used by Nussenzveig and Wiscombe, in line with the more commonly used convention of van de Hulst.[2]] In writing Eq. (1), we have corrected an error that crept into Ref. 1.[3] [This misprint was subsequently corrected.[4]]

Q_{ext} is related to the forward-scattering amplitude through the optical theorem[2]

$$Q_{\text{ext}} = \frac{4}{x^2}\,\text{Re}\,S(0). \tag{2}$$

Because of the relation between the scattered electric field and the scattering amplitude, plus the fact that the scattered field must be real, it is easy to show[5] that S must obey a crossing condition, namely,

$$S(0, -k) = [S(0, k)]^*. \tag{3}$$

Based on Eqs. (1) and (2) we now make the quite plausible assumption that the asymptotic expansion for $S(0)$ may be written as

$$\frac{4}{x^2}S(0) = 2 + 1.9923861\,(1+i\alpha)x^{-2/3} + i\frac{8}{x}\left\{\frac{m^2+1}{4(m^2-1)^{1/2}} - F\exp[-2i(m-1)x] - \sum_{j=1}^{\infty}\left(\frac{m-1}{2j+1-m}\right)\left(\frac{m-1}{m+1}\right)^{2j}\times\exp[-2i(m-1+2jm)x]\right\} - 0.7153537(1+i\beta)\,x^{-4/3} - i0.3320643\,e^{-i\pi/3}\times\frac{m^2+1}{(m^2-1)^{3/2}}(2m^4-6m^2+3)x^{-5/3} + 0(x^2) + \text{ripple}, \tag{4}$$

where α and β are (real) constants to be determined.

Since $x = kr$, the crossing condition on S may also be written as

$$S(0, -x) = [S(0, x)]^*, \tag{5}$$

i.e.,

$$\text{Re}\,S(0, -x) = \text{Re}\,S(0, x), \tag{6a}$$

$$\text{Im}\,S(0, -x) = -\text{Im}\,S(0, x). \tag{6b}$$

It is easy to see that, with the exception of the terms of order $x^{-2/3}$ and $x^{-4/3}$, all terms in Eq. (4) obey these conditions. [We should point out that in the published version of Eq. (1) (see Ref. 1), a missing factor of x in the summation term meant that it did not satisfy Eq. (5): The corrected

0740-3232/86/020256-03$02.00

version does.] We now consider the term in $x^{-2/3}$. Since in our sign convention we require S to be analytic in the lower half-plane, we note that

$$(-x)^{-2/3} = e^{2\pi i/3}x^{-2/3} = (-1/2 + i1/2\sqrt{3})x^{-2/3}. \tag{7}$$

Thus crossing for this term becomes

$$\begin{aligned}(1 + i\alpha)(-x)^{-2/3} &= 1/2(1 + i\alpha)\,(-1 + i\sqrt{3})x^{-2/3} \\ &= (1 - i\alpha)x^{-2/3}. \end{aligned} \tag{7'}$$

This equation has the solution

$$\alpha = -\sqrt{3}. \tag{8}$$

In the case of the term in $x^{-4/3}$, we require that

$$(-x)^{-4/3} = e^{4\pi i/3}x^{-2/3} = -1/2(1 + i\sqrt{3})x^{-4/3} \tag{9}$$

so that

$$\beta = \sqrt{3}. \tag{10}$$

Having evaluated α and β, we propose the following asymptotic expansion for $S(0)$:

$$\begin{aligned}\frac{4}{x^2}S(0) = 2 &+ 1.9923861(1 - i\sqrt{3})x^{-2/3} \\ &+ i\frac{8}{x}\left\{\frac{m^2+1}{4(m^2-1)^{1/2}} - F\exp[-2i(m-1)x]\right. \\ &- \sum_{j=1}^{\infty}\left(\frac{m-1}{2j+1-m}\right)\left(\frac{m-1}{m+1}\right)^{2j} \\ &\left.\times \exp[-2i(m-1+2jm)x]\right\} \\ &- 0.7153537(1 + i\sqrt{3})x^{-4/3} - i0.3320643e^{-i\pi/3} \\ &\times \frac{m^2+1}{(m^2-1)^{3/2}}(2m^4 - 6m^2 + 3)x^{-5/3} \\ &+ 0(x^{-2}) + \text{ripple}. \end{aligned} \tag{11}$$

In his classic book, van de Hulst[2] examined scattering by large particles from several perspectives. The first term (i.e., 2) is due to diffraction, and the exponential term is due to nearly central rays, while the sum of exponentials represent multiple internal reflections of these near central rays.[6] The term in $x^{-2/3}$ is evidently due to edge effects,[7] with van de Hulst being able to obtain the result

$$\frac{4}{x^2}S^{\text{edge}}(0) = 1.84(1 - i\sqrt{3})x^{-2/3} \tag{12}$$

in the case of totally reflecting bodies. (Note that S^{edge} will always obey the crossing condition, as a direct result of its formulation.) For more normal refractive indices, van de Hulst was forced to rely on empirical fits. For $m = 1.33$, he obtained.

$$\frac{4}{x^2}S^{\text{edge}}(0, 1.33) = 1.84(1 - i0.78\sqrt{3})x^{-2/3}. \tag{13}$$

The discrepancy in the imaginary term can be largely traced to the lack of sufficient published data on the imaginary part of $S(0)$ at that time. In the case of $m = 1.20$, he was able to obtain only the real coefficient, which he determined to be 2.04. Note that a further source of error in the imaginary term is the absence of the term

$$i\frac{2}{x}\frac{m^2+1}{(m^2-1)^{1/2}} \cong i6.3x^{-1}$$

in the case of $m = 1.33$, in van de Hulst's formulation.

Recently, Senior[8] applied complex-angular-momentum theory to the special case of a perfectly conducting sphere. His result may be written as

$$S(0) = S^0 + S^{c(1)} + S^{c(2)}, \tag{14}$$

where

$$\frac{4}{x^2}S^0 = 2 - \frac{11}{6}x^{-2} \tag{14a}$$

and

$$\begin{aligned}\frac{4}{x^2}S^{c(2)} &= (0.331888 - i0.576076)x^{-2/3} + 0(x^{-4/3}) \\ &= 0.331888(1 - i1.002138\sqrt{3})x^{-2/3}, \end{aligned} \tag{14b}$$

and $S^{c(1)}$ "decreases exponentially with increasing x."[8] Again we see that, to within a probable numerical uncertainty, $S^{c(2)}$ obeys the crossing condition. (In fact, crossing may be used to test the accuracy of the six-digit numbers cited in Ref. 8.)

We have investigated the accuracy of Eq. (11) for a series of refractive indices consisting of real parts of 1.1, 1.33, 1.5, and 2.0 and imaginary parts of 0.0, −0.01, −0.1, and −0.5. We have concluded from this study that the absolute accuracy of the imaginary part of Eq. (11) is comparable with that of the real part, which has been thoroughly investigated by Nussenzveig and Wiscombe.[1] However, since the imaginary part is clearly significantly smaller than the real part (by 1 to 2 orders of magnitude for x between 100 and 1000), it is clear that the fractional error in the imaginary part is much larger than that of the real part. We also note that, in the case of a real refractive index, the ripple structure of the true Mie amplitude is not accounted for by this approximation.[1,9]

It would be possible to add to Eq. (11) a term of the form $i \times$ constant $\times\, x^{-1}$, which will obey crossing, provided that this constant is real. A careful study of the results that we have obtained suggests that, if such a term does exist, the constant involved can be no larger than about 0.02. It is thus doubtful that such a term exists, and we believe that Eq. (11), as it stands, is the correct asymptotic form for $S(0)$.

In a subsequent paper,[10] we perform some contour integrals on $S(0)$ and derive a number of sum rules. The numerical accuracy of these results provides further support for the validity of Eq. (11).

It has been pointed out by one reviewer that our result is partially contained in the Ph.D. dissertation of Khare.[11] In particular, the complex coefficients of $x^{-2/3}$ and $x^{-4/3}$ are given on p. 108 of Ref. 11. The phases of these coefficients are the same as for scalar scattering, which has already been treated at length for real refractive indices.[12] We thank the reviewer for bringing these references to our attention.

REFERENCES

1. H. M. Nussenzveig and W. J. Wiscombe, "Efficiency factors in Mie scattering," Phys. Rev. Lett. **45**, 1490–1494 (1980). (See also W. J. Wiscombe and H. M. Nussenzveig, "Replacing Mie scattering calculations with asymptotic formulae," presented at the International Radiation Symposium 1980, Fort Collins, Colo., August 11–16, 1980).
2. H. C. van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981).
3. W. J. Wiscombe, NASA Goddard Space Flight Center, Greenbelt, Md. 20771 (personal communication).
4. H. M. Nussenzveig, "M. Schönberg on his 70th birthday," Rev. Bras. Fis. (Special Volume), 302–319 (1984).
5. M. A. Box and B. H. J. McKellar, "Long-wavelength limit of scattering from a lossy dielectric sphere," J. Opt. Soc. Am. **72**, 1090–1091 (1982).
6. Ref. 2, Sec. 12.45.
7. Ref. 2, Sec. 17.26.
8. T. B. A. Senior, "Note on the extinction efficiency," Appl. Opt. **22**, 1796–1797 (1983).
9. H. M. Nussenzveig, and W. J. Wiscombe, "Forward optical glory," Opt. Lett.**5**, 455–457 (1980).
10. M. A. Box, P. Attard, G. Bryant, and B. H. J. McKellar, "Sum rules for Mie scattering," J. Opt. Soc. Am. A (to be published).
11. V. Khare, "Short-wavelength scattering of electromagnetic waves by a homogeneous dielectric sphere," Ph.D. dissertation (University of Rochester, Rochester, N.Y., 1976).
12. H. M. Nussenzveig, "High-frequency scattering by a transparent sphere. I. Direct reflection and transmission," J. Math. Phys. **10**, 82–124 (1969) (see especially Subsec. 4 D).