General perturbation technique for the calculation of radiative effects in scattering and absorbing media

Igor N. Polonsky and Michael A. Box

School of Physics, University of New South Wales, Sydney, NSW 2052, Australia

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Recently it has been shown that the perturbation technique, based on joint use of both the direct and the adjoint solutions of the radiative transfer equation, is a powerful tool to solve and analyze various time-independent one-dimensional problems of atmospheric physics such as the calculation of weighting functions, prediction of radiative effects, and development of retrieval algorithms. Our primary goal is to obtain a general formulation of the perturbation technique for the most general case of the radiative transfer problem: time-dependent problems, with regard to polarization, and any possible external sources of radiation such as laser beams and solar illumination. Possible areas of application of the perturbation technique are discussed, and several examples to illustrate them are provided. The accuracy of this technique is discussed by considering the particular examples.

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1. INTRODUCTION

The propagation of electromagnetic radiation through scattering media is widely used to obtain information about the media (by use of satellite remote sensing and lidar sounding), to observe through such a medium (by use of satellite imaging systems, underwater television, and tissue tomography), or to establish an optical communication channel. For all these applications, and many others, the vector radiative transfer equation (VRTE)[1] is used to describe such propagation. Even in the relatively simple case of solar illumination of a plane-parallel atmosphere, as well as substantially more complex problems of three-dimensional (3-D) radiative transfer or laser pulse propagation, various analytical approximations and numerical techniques to solve the VRTE have constantly been developed and refined. The difficulties (and time requirements) to obtain either a numerical or an analytical solution increase dramatically with increases in the variable space dimension, i.e., from one dimension to three dimensions; adding temporal dependence; taking into account polarization effects; or any combination of these.

The necessity to solve the VRTE is the main reason that many numerical and analytical techniques have been developed.[1–3] Many were inherited from nuclear reactor theory[4] because some of the problems of neutron transport are almost identical to the corresponding radiation transfer problems. However, as a rule, all these techniques provide solutions with a limited area of applicability. For example, the small-angle approximation has been developed to describe radiative transfer through an absorbing and scattering medium with an extended phase function. In contrast, the diffusion approximation allows one to calculate radiation characteristics in nonabsorbing (or weakly absorbing) optically thick scattering media. Similar limits for numerical techniques may be given and can be found in corresponding books and papers (see above).

Can we reuse existing solution methods of certain radiative transfer problems beyond the strict area of their applicability to address more complex problems? The perturbation technique makes this possible. The idea of perturbation is not new and has a long history of use in numerous areas, not only in radiative transfer theory but also in many areas of mathematical physics. The basic idea is simple. If our system is governed by a linear matrix, integral, or differential equation, then a sufficiently small variation (nonsingular in character) of the system parameters results in a corresponding small perturbation of the system response. In other words, if our complex problem under consideration closely resembles one that is simpler and has an appropriate solution, there is a way to use such a simple solution to solve the more complex problem.

There are several different ways to construct a perturbation theory, general ideas of which are discussed in the literature.[5,6] One way is based on the introduction of small parameters into the VRTE and then exploring this idea by taking into account particular features of a method to solve the chosen radiative transfer equation. This approach was used by Romanova[7] to build a mathematical model of the effect of horizontal inhomogeneities on radiance propagation. The same technique, but within the framework of the diffusion approximation, was used by Li *et al.*[8,9] to investigate the effect of a small sine-shaped horizontal variation of the extinction coefficient on the characteristics of the upwelling and downwelling radiation. Recently it was used to find the weighting functions for the purpose of remote-sensing retrieval of atmospheric constituents in the study by Spurr *et al.*[10] However, the implementation of this approach assumes that all possible types of perturbation are known, so that each new type of the perturbation requires a complete reconsideration of the problem from the very beginning.

A second way to construct a perturbation technique is based on the joint use of both the direct and the adjoint solutions of the same problem. Since it does not depend on the particular technique to solve the unperturbed equation, this approach allows one to obtain results in a more general form than the first one does. This perturbation technique was inherited from neutron transport theory and suggested for use in radiation transfer applications by Marchuk[11] and independently by Gerstl.[12] It was found useful in developing an efficient general technique to calculate radiative effects in a series of papers by Box *et al.*[13–16] Ustinov applied this approach to analytic studies of the case of thermal radiation in a purely absorbing atmosphere,[17] and also with the inclusion of scatterers,[18] and in the case of solar radiation reflected from a scattering, optically thick vertically inhomogeneous planetary atmosphere.[19–21] It was also applied to the ozone retrieval problem from passive satellite sounding by Landgraf *et al.*[22,23] It was shown that this approach is an accurate tool for analyzing the sensitivity of upwelling radiation to aerosol composition for typical satellite optical remote-sensing problems either for a homogeneous medium[24] or for a stratified slab.[25] Moreover, this idea allows one to obtain a solution even if radiance polarization is taken into account.[26–28]

The joint use of the adjoint and direct solutions to construct the perturbation theory is fully equivalent to application of the Green function theory, in which employing the reciprocity relations allows one to obtain the same final result. In fact, the reciprocity of the Green function and its adjoint[4] is the key to all reciprocity relations.[13] This approach was utilized to develop image transfer theory,[3] to obtain an analytical solution for a lidar return with regard to multiple scattering and polarization,[29] and to find the weighting functions in satellite remote sensing.[30]

The main purpose of this paper is to provide general equations relevant to the perturbation calculation on the basis of the joint solution of the direct and adjoint problems, which can be used in various applications of scattering medium optics. The advantages of this perturbation technique in analyzing a sophisticated problem by means of reducing it to a simpler one will be demonstrated for selected problems of radiative transfer theory. The paper's organization is as follows. Section 2 introduces the basic notation and equations that are necessary to describe the time-dependent polarized radiative transfer problem based on the Stokes vector framework. The details of the adjoint formulation of the radiative transfer problem—equations, boundary conditions, and the relationship of the solution of the adjoint problem to the equivalent direct problem— are discussed in Section 3. Section 4 is devoted to the derivation of the basic formulas of the perturbation approach and consideration of its advantages and limitations. Section 5 shows the application of the perturbation approach to two sample problems: polarized lidar sounding and 3-D radiative transfer. The accuracy of the perturbation prediction is discussed on the basis of comparison with results of a fully numerical simulation of the same problems.

2. VECTOR RADIATIVE TRANSFER EQUATION: DIRECT FORMULATION

To characterize both the energy and the polarization properties of light at the point $\mathbf{r}$ in the direction $\mathbf{n}$ at time t, the real Stokes vector $\mathbf{I} = \{I, Q, U, V\}$ is mostly used. There are several definitions of the Stokes vector, which differ in the choices of coordinate system and of what direction of rotation is defined as positive. Some discussion of this problem can be found in the paper by Hovenier and van der Mee.[31] Our definition is the same as that of Chandrasekhar.[31,32] Taking into account the existence of the complex representation of the Stokes vector introduced by Kuščer and Ribarič,[33] we shall assume that the Stokes vector may consist of complex numbers and use the terms real and complex to underline which particular representation is employed. Additionally, we will use capital letters to denote the Stokes vectors and matrices to distinguish them from the geometrical vectors, which will be denoted by lower-case letters.

Consider polarized light propagation in a scattering and absorbing medium, bounded by a convex surface $S(\mathbf{r})$. In this case, the equation of radiative transfer may be written as

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{n}\cdot\nabla + \sigma_e(\mathbf{r})\right]\mathbf{I}(t, \mathbf{r}, \mathbf{n}) = \frac{\sigma_s(\mathbf{r})}{4\pi}\int_{4\pi}\mathbf{Z}(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{I}(t, \mathbf{r}, \mathbf{n}')\mathrm{d}\mathbf{n}' + \mathbf{Q}(t, \mathbf{r}, \mathbf{n}). \quad (1)$$

Here ∇ is the gradient with respect to $\mathbf{r}$, $\sigma_e(\mathbf{r})$ and $\sigma_s(\mathbf{r})$ are the extinction and scattering coefficients at point $\mathbf{r}$, respectively, $\mathbf{I}(t, \mathbf{r}, \mathbf{n})$ is the Stokes vector of the radiance at point $\mathbf{r}$ in the direction $\mathbf{n}$ at time t, $\mathbf{Z}(\mathbf{r}, \mathbf{n}, \mathbf{n}')$ is the phase matrix, and $\mathbf{Q}(t, \mathbf{r}, \mathbf{n})$ represents all sources of radiation, including a laser pulse, solar radiation, or internal emission sources. As discussed by Chandrasekhar,[32] the phase matrix is related to the scattering matrix $\mathbf{P}(\mathbf{r}, \cos\beta)$ by

$$\mathbf{Z}(\mathbf{r}, \mathbf{n}, \mathbf{n}') = \mathbf{L}(\pi - \chi_1)\mathbf{P}(\mathbf{r}, \cos\beta)\mathbf{L}(-\chi_2). \quad (2)$$

Here β is the scattering angle and $\mathbf{L}(\chi)$ is the matrix that is required to rotate a meridian plane through angles $\pi - \chi_1$ and $-\chi_2$ before and after scattering, onto a local scattering plane. The definitions of χ_1 and $-\chi_2$ are exactly the same as those used by Hovenier and van der Mee,[31] and their particular forms are not important for the purposes of this discussion. For our further consideration, it is necessary to make no assumption about the phase matrix except that it satisfies an important reciprocity relation:[34]

$$\mathbf{Z}(\mathbf{r}, -\mathbf{n}', -\mathbf{n}) = \mathbf{T}\mathbf{Z}^{+}(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{T}. \quad (3)$$

Here $(+)$ denotes the Hermitian conjugation operation. The explicit form of the matrix $\mathbf{T}$ depends on the Stokes vector representation, and for the real case

$$\mathbf{T} = \mathrm{diag}[1, 1, -1, 1]. \quad (4)$$

Equation (3) is a consequence of the symmetry of light-scattering processes with respect to inversion of time and follows directly from Maxwell's equations for any particle in an arbitrary orientation, assuming that only the dielectric, permeability, and conductivity tensors are symmetric. Note that the scattering matrix of an optically isotropic scatterer has the form[35]

$$\mathbf{P}(\mathbf{r}, \cos\theta) = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{bmatrix}, \tag{5}$$

with normalization

$$\frac{1}{2}\int_0^{\pi} a_1(\cos\beta)\sin\beta \mathrm{d}\beta = 1. \tag{6}$$

To complete Eq. (1), it is usually required that its solution should obey appropriate boundary and initial conditions. The initial condition has the form

$$\mathbf{I}(t, \mathbf{r}, \mathbf{n}) = 0 \qquad \text{for } t \leqslant t_1. \tag{7}$$

The boundary condition is formulated on the basis of the principle that there is no radiation entering the medium from outside other than the sources, $\mathbf{Q}$. In the case of a nonreflective boundary surface, the boundary condition has the simplest form of

$$\mathbf{I}(t, \mathbf{r}, \mathbf{n}) = 0 \qquad \text{for } \mathbf{r} \in S(\mathbf{r}), \qquad \mathbf{n} \cdot \mathbf{n}_{\perp} > 0, \tag{8}$$

where $\mathbf{n}_{\perp}$ is the normal to the boundary surface, which is directed into the medium. The reflection properties of the boundary surface can be taken into account by means of an appropriate modification of the right-hand side of Eq. (8):

$$\mathbf{I}(t, \mathbf{r}, \mathbf{n}) = \frac{1}{\pi}\int_{\mu'<0} \mathbf{M}(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{I}(t, \mathbf{r}, \mathbf{n}')\mu' \mathrm{d}n' \qquad \text{for } \mathbf{r} \in S(\mathbf{r}), \quad \mathbf{n} \cdot \mathbf{n}_{\perp} > 0, \tag{9}$$

where $\mu' = \mathbf{n}_{\perp} \cdot \mathbf{n}'$ and $\mathbf{M}(\mathbf{r}, \mathbf{n}, \mathbf{n}')$ is the matrix that determines the reflection properties of the boundary surface. In the simplest case of Lambertian reflection, $\mathbf{M}(\mathbf{r}, \mathbf{n}, \mathbf{n}')$ has the form

$$\mathbf{M}(\mathbf{r}, \mathbf{n}, \mathbf{n}') = \begin{bmatrix} A(\mathbf{r}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

Here $A(\mathbf{r})$ is the coefficient for Lambertian reflection at the position $\mathbf{r}$.

To simplify the necessary manipulations, it is more convenient to use an operator notation. Equation (1) takes the form

$$\hat{\mathbf{L}}\mathbf{I}(t, \mathbf{r}, \mathbf{n}) = \mathbf{Q}(t, \mathbf{r}, \mathbf{n}), \tag{11}$$

where the transport operator $\hat{\mathbf{L}}$ is clearly defined as

$$\hat{\mathbf{L}} = \frac{1}{c}\frac{\partial}{\partial t} + \mathbf{n} \cdot \nabla + \sigma_e(\mathbf{r}) - \frac{\sigma_s(\mathbf{r})}{4\pi}\int_{4\pi} \mathrm{d}\mathbf{n}'\mathbf{Z}(\mathbf{r}, \mathbf{n}, \mathbf{n}') \circ. \tag{12}$$

(Note that the notation $\circ$ is used to indicate that the final term is an integral operator, not merely a definite integral.)

3. DEFINITION OF THE ADJOINT OPERATOR, ITS EXPLICIT FORM, AND THE BOUNDARY CONDITIONS FOR THE ADJOINT EQUATION

Let us introduce a scalar product $\langle \mathbf{G}^+, \mathbf{H}\rangle$ of two vectors,

$$\mathbf{G}(t, \mathbf{r}, \mathbf{n}) = \begin{pmatrix} G_I(t, \mathbf{r}, \mathbf{n}) \\ G_Q(t, \mathbf{r}, \mathbf{n}) \\ G_U(t, \mathbf{r}, \mathbf{n}) \\ G_V(t, \mathbf{r}, \mathbf{n}) \end{pmatrix},$$

$$\mathbf{H}(t, \mathbf{r}, \mathbf{n}) = \begin{pmatrix} H_I(t, \mathbf{r}, \mathbf{n}) \\ H_Q(t, \mathbf{r}, \mathbf{n}) \\ H_U(t, \mathbf{r}, \mathbf{n}) \\ H_V(t, \mathbf{r}, \mathbf{n}) \end{pmatrix}, \tag{13}$$

as the following integral over the entire range, Ξ, of the problem variables t, $\mathbf{r}$, $\mathbf{n}$:

$$\begin{aligned} \langle \mathbf{G}^+, \mathbf{H}\rangle &= \int\int\int_{\Xi} \mathbf{G}^+\mathbf{H}\mathrm{d}\Xi \\ &= \int\int\int_{\Xi} \sum_j \bar{G}_j(t, \mathbf{r}, \mathbf{n}) \\ &\quad \times H_j(t, \mathbf{r}, \mathbf{n})\mathrm{d}t\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{n}, \qquad j = I, Q, U, V. \end{aligned} \tag{14}$$

Here $\bar{G}$ denotes the complex conjugation of G. The main reason for this choice of the scalar product is that the result of an optical measurement may be represented in such a form. Moreover, it is invariant with respect to orthogonal transformations of the Stokes vector parameters. In particular, it is invariant with respect to transfer to the complex representation of the Stokes vector.[31]

We define the adjoint operator, $\hat{\tilde{\mathbf{L}}}$, by requiring that for all suitable $\mathbf{G}$ and $\mathbf{H}$

$$\langle \mathbf{G}^+, \hat{\mathbf{L}}\mathbf{H}\rangle = \langle (\hat{\tilde{\mathbf{L}}}\mathbf{G})^+, \mathbf{H}\rangle. \tag{15}$$

Let us introduce a set of functions $\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n})$ that satisfy the adjoint equation

$$\hat{\tilde{\mathbf{L}}}\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n}) = \tilde{\mathbf{Q}}(t, \mathbf{r}, \mathbf{n}), \tag{16}$$

where $\tilde{\mathbf{Q}}(t, \mathbf{r}, \mathbf{n})$ represents the adjoint sources. Naturally, $\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n})$ also should satisfy some boundary and initial conditions, which depend on the explicit form of the adjoint equation. Following Bell and Glasstone[4] and introducing the necessary generalization, we find that it is simpler and physically clearer to define the adjoint operator $\hat{\tilde{\mathbf{L}}}$ in the form

$$\hat{\tilde{\mathbf{L}}} = -\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{n} \cdot \nabla + \sigma_e(\mathbf{r}) - \frac{\sigma_s(\mathbf{r})}{4\pi}\int_{4\pi} \mathrm{d}\mathbf{n}'\mathbf{T}\mathbf{Z}(\mathbf{r}, -\mathbf{n}, -\mathbf{n}')\mathbf{T}\circ \tag{17}$$

and then to derive the corresponding initial and boundary conditions. This can be done by substituting Eqs. (12)

and (17) into Eq. (15). After several transformations (see Appendix A for more details), we come to the following equation:

$$\frac{1}{c}\iint [\tilde{\mathbf{I}}^{+}(t_2, \mathbf{r}, \mathbf{n})\mathbf{I}(t_2, \mathbf{r}, \mathbf{n}) - \tilde{\mathbf{I}}^{+}(t_1, \mathbf{r}, \mathbf{n})\mathbf{I}(t_1, \mathbf{r}, \mathbf{n})]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{n} + \iint \left[\int_S \tilde{\mathbf{I}}^{+}(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})(\mathbf{n}\cdot\mathbf{n}_{\perp})\mathrm{d}S\right]\mathrm{d}t\mathrm{d}\mathbf{n} = 0. \quad (18)$$

Here $t_2 > t_1$ is some arbitrary moment of time. This allows us to deduce the boundary conditions for the solution of the adjoint problem. Actually, by taking into account the boundary and initial conditions for $\mathbf{I}(t, \mathbf{r}, \mathbf{n})$ and the fact that $\mathbf{I}(t, \mathbf{r}, \mathbf{n})$ and $\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n})$ are both completely arbitrary and independent functions of t, $\mathbf{r}$, $\mathbf{n}$, one can easily check that Eq. (15) is turned into an equality if and only if $\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n})$ is subject to initial and boundary conditions of the form

$$\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n}) = 0 \qquad \text{for } t \geqslant t_2, \quad (19)$$

$$\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n}) = -\frac{1}{\pi}\int_{\mu'>0}\mathbf{TM}(\mathbf{r}, -\mathbf{n}, -\mathbf{n}')\mathbf{T}\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n}')\mu'\mathrm{d}\mathbf{n}' \qquad \text{for } \mathbf{r} \in S(\mathbf{r}), \quad \mathbf{n}\cdot\mathbf{n}_{\perp} < 0. \quad (20)$$

A direct comparison of the explicit form of the transfer operators and the corresponding initial and boundary conditions in the direct and adjoint formulations show that

$$\hat{\tilde{\mathbf{L}}}(t, \mathbf{r}, \mathbf{n}, \mathbf{n}') = \mathbf{T}\hat{\mathbf{L}}(-t, \mathbf{r}, -\mathbf{n}, -\mathbf{n}')\mathbf{T}, \quad (21)$$

$$\tilde{\mathbf{I}}(t, \mathbf{r}, \mathbf{n}) = \mathbf{T}\mathbf{I}(-t, \mathbf{r}, -\mathbf{n}) \quad (22)$$

if

$$\tilde{\mathbf{Q}}(t, \mathbf{r}, \mathbf{n}) = \mathbf{T}\mathbf{Q}(-t, \mathbf{r}, -\mathbf{n}). \quad (23)$$

Equation (22) means that the solution of a given adjoint problem is equivalent to the solution of the corresponding direct problem with the effective source [Eq. (23)]. In other words, to obtain the solution of the adjoint and direct problems we need to solve only the direct problem but for two sources. Note that some techniques, such as the discrete-ordinate[36] or the spherical harmonics approximation,[37] allow one to make simulations for several sources at the same time. Furthermore, it has recently been implemented in the DISORT framework.[38]

4. PERTURBATION APPROACH

As a rule, an optical measurement or a radiation effect E (for the sake of simplicity, we will label both as an effect in this paper) may be represented in the form of the above-defined scalar product of a suitable response function $\mathbf{R}(t, \mathbf{r}, \mathbf{n})$ and the Stokes vector of the radiance $\mathbf{I}(t, \mathbf{r}, \mathbf{n})$:

$$E = \langle \mathbf{R}^{+}, \mathbf{I}\rangle. \quad (24)$$

For example, to describe the flux through the plane $z = z_0$, we can use the following receiver function:

$$\mathbf{R}(t, \mathbf{r}, \mathbf{n}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\delta(z - z_0)\cos\theta, \quad (25)$$

where θ is the angle between the direction $\mathbf{n}$ and the z axis. Similarly, a measurement of the circular polarization component at position $\mathbf{r}_0$ in the direction $\mathbf{n}_0$ at time t_0 can be represented by Eq. (24) with the receiver function

$$\mathbf{R}(t, \mathbf{r}, \mathbf{n}) = \begin{pmatrix}0.5\\0\\0\\0.5\end{pmatrix}\delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{n} - \mathbf{n}_0)\delta(t - t_0). \quad (26)$$

The definition of the adjoint problem [Eq. (16)] and the relationship [Eq. (15)] allow us to rewrite Eq. (24) in the form

$$E = \langle \mathbf{R}^{+}, \mathbf{I}\rangle = \langle(\hat{\tilde{\mathbf{L}}}\tilde{\mathbf{I}})^{+}, \mathbf{I}\rangle = \langle\tilde{\mathbf{I}}, \hat{\mathbf{L}}\mathbf{I}\rangle = \langle\tilde{\mathbf{I}}^{+}, \mathbf{Q}\rangle. \quad (27)$$

This means that there are two independent ways to calculate the same effect by use of the solution of either the direct or the adjoint problem. Which way should be chosen? The answer depends on what problem is under consideration. Because there is no mathematical difference to implement either the direct or the adjoint formulation, the only common recommendation is to use the more computationally efficient approach. However, some techniques successfully use both approaches. For example, during a Monte Carlo calculation for a nonabsorbing medium, a problem arises as to when is it possible to stop tracking the current photon. To solve this problem, we have to estimate the contribution of that photon to the receiver signal, which is determined by the function $\mathbf{I}^{+}(t, \mathbf{r}, \mathbf{n})$. Note that even an approximate solution of the corresponding adjoint problem is quite enough to make such a decision.[39]

Assuming we have the solutions to both the direct and the adjoint problems, we may construct the perturbation theory. Let us consider two media that have the same geometrical shape but whose scattering and absorbing characteristics, as well as the reflection properties of the boundary surface, may differ a little; we may characterize the media by

$$\hat{\mathbf{L}}_p = \hat{\mathbf{L}}_b + \delta\hat{\mathbf{L}}, \qquad \hat{\tilde{\mathbf{L}}}_p = \hat{\tilde{\mathbf{L}}}_b + \delta\hat{\tilde{\mathbf{L}}}, \quad (28)$$

where $\hat{\mathbf{L}}_b$ and $\hat{\mathbf{L}}_p$ are the transport operators for the first medium (we call it the base medium) and the second medium (perturbed), respectively. Similar notation is used for the corresponding adjoint operators $\hat{\tilde{\mathbf{L}}}_b$ and $\hat{\tilde{\mathbf{L}}}_p$. Further, we will use indexes b and p to denote the operators and radiances corresponding to the base and perturbed cases, respectively.

Let us consider the variation δE of the effect E:

$$\delta E = \langle \mathbf{R}^{+}, \mathbf{I}_p\rangle - \langle \mathbf{R}^{+}, \mathbf{I}_b\rangle. \quad (29)$$

After straightforward manipulation (see Appendix B for details), we obtain the basic formula of the perturbation approach

$$\delta E = -\langle \tilde{\mathbf{I}}_b^+,\ \delta\tilde{\mathbf{L}}\mathbf{I}_b\rangle + \langle \delta\tilde{\mathbf{I}}^+,\ \hat{\mathbf{L}}_p\delta\mathbf{I}\rangle. \tag{30}$$

Note that we have made no assumptions to derive this result, and hence it is valid for the most general case. The right-hand side of Eq. (30) consists of two terms. From the physical point of view, it is clear that the first term outlines the effect of medium perturbation, which does not change the radiance noticeably, and the second term provides us with an estimation of the effect of the radiance variation. Having the direct and adjoint solutions of the base problem, we can calculate the first term easily, whereas to calculate the second term we need to know both $\delta\tilde{\mathbf{I}}$ and $\delta\mathbf{I}$, which might require an unacceptable amount of effort and computer time, which is usually undesirable. However, if we assume that for any admissible $\mathbf{F}$

$$\|\delta\hat{\mathbf{L}}\| = \sup_{\|\mathbf{F}\|\neq 0}\frac{\|\delta\hat{\mathbf{L}}\mathbf{F}\|}{\|\mathbf{F}\|} \leqslant \epsilon \ll 1, \tag{31}$$

where $\|\mathbf{F}\| = \sqrt{\langle \mathbf{F}^+,\ \mathbf{F}\rangle}$ is the norm of $\mathbf{F}$ in the corresponding Hilbert space, then

$$\|\langle \tilde{\mathbf{I}}_b^+,\ \delta\tilde{\mathbf{L}}\mathbf{I}_b\rangle\| \leqslant \|\tilde{\mathbf{I}}_b^+\|\,\|\delta\hat{\mathbf{L}}\mathbf{I}_b\| \leqslant \epsilon\|\tilde{\mathbf{I}}_b^+\|\,\|\mathbf{I}_b\|, \tag{32a}$$

$$\|\langle \delta\tilde{\mathbf{I}}^+,\ \hat{\mathbf{L}}_p\delta\mathbf{I}\rangle\| \leqslant \|\delta\tilde{\mathbf{I}}^+\|\,\|\hat{\mathbf{L}}_p\delta\mathbf{I}\| = \|\delta\tilde{\mathbf{I}}^+\|\,\|\delta\hat{\mathbf{L}}\mathbf{I}_b\|$$
$$\leqslant \epsilon^2\frac{\|\tilde{\mathbf{I}}_b^+\|\,\|\mathbf{I}_b\|}{\|\hat{\mathbf{L}}_p\|}. \tag{32b}$$

Note that our norm definition has an obvious property of $\|\mathbf{F}\| = \|\mathbf{F}^+\|$ and $\|\hat{\mathbf{L}}\| = \|\hat{\tilde{\mathbf{L}}}\|$. Therefore if ϵ is small enough, the second term of Eq. (30) can be neglected as it is of the second order with respect to ϵ, and we finally obtain

$$\delta E \approx -\langle \tilde{\mathbf{I}}_b^+,\ \delta\hat{\mathbf{L}}\mathbf{I}_b\rangle. \tag{33}$$

This equation tends to be more accurate the smaller ϵ is.

In many cases we may write

$$\delta\hat{\mathbf{L}} = \frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}p}\delta p, \tag{34}$$

where p is some parameter (for example, the particle density or the refractive index). Then expression (33) provides us with the estimation of the corresponding derivative

$$\frac{\mathrm{d}E}{\mathrm{d}p} = -\left\langle \tilde{\mathbf{I}}_b^+, \frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}p}\mathbf{I}_b\right\rangle. \tag{35}$$

These two formulas [approximation (33) and Eq. (35)] are the foundation of the perturbation technique and determine two main directions of its possible application. First, we may estimate the magnitude of the effect variation that is due to a given small perturbation of the base medium. This is the way that the perturbation theory was used by Box *et al.*[15] to estimate the effect of aerosol model variation on fluxes. Another good example of such an application is image transfer theory, as we can treat an object under observation as a medium perturbation, which is admissible in problems of target detection and identification and of optical medical imaging. We shall discuss in Section 5 how the perturbation theory can be used to describe the lidar signal reflected from an optically dense cloud and radiance propagation through a horizontally inhomogeneous medium.

Another use of the perturbation theory is to study how sensitive the effect E is to a medium perturbation of a given kind. In the theory of remote sensing this derivative is usually called the weighting function. Examples of such a calculation for some problems of passive satellite remote sensing can be found in studies by Ustinov[19,28] and by Rozanov *et al.*[30] Note that Eq. (35) provides a computationally efficient and accurate way to estimate such derivatives. The use of the finite-difference scheme to estimate these derivatives requires that solutions for two different media be obtained, whereas the perturbation approach requires one to obtain the direct and adjoint solutions for the same medium. The situation becomes even more favorable for the perturbation technique when sensitivity to more than one parameter is under investigation.

To finalize discussion of Eq. (35), let us compare it with the way to calculate the weighting function used by Spurr *et al.*[10] They employed a combination of the idea of smallness of some parameters in the radiative transfer equation with the discrete-ordinate approach for its solution. This technique allows them to deduce a rather complicated but efficient scheme for calculating the weighting functions. The advantage of their approach is that it allows them to study how a given variation of the medium characteristic affects the radiance at any height and in any direction. Unfortunately, this solution cannot be reused if the unperturbed problem is governed by a more complicated radiative transfer equation, whereas Eq. (35) has no such limitation. Additionally, taking into account that for most practical situations there is one height at which the radiance variation is of interest (the receiver altitude), the use of Eq. (35) seems to provide a simpler way to obtain the weighting functions.

In general, we cannot *a priori* presume high accuracy for approximation (33) because to derive it we have introduced some error by neglecting $\langle \delta\tilde{\mathbf{I}}^+,\ \hat{\mathbf{L}}_p\delta\mathbf{I}\rangle$. Relations (32), which provide a formal mathematical condition, are not useful for realistic estimation. However, by analyzing the structure of the neglected term we note that it is determined by the variations of $\delta\tilde{\mathbf{I}}^+$ and $\delta\mathbf{I}$, which are generated by introducing the perturbation $\delta\hat{\mathbf{L}}$. Therefore if a given medium perturbation does not change the radiance field significantly, we may expect high accuracy from approximation (33). Although it is a qualitative criterion and there is no direct recipe to check it for all possible cases, it allows one to detect some bad cases by undertaking careful and rigorous physical analysis of a given problem. Various approximations and asymptotic solutions[1–3] may make such an analysis simpler.

5. SELECTED APPLICATIONS OF THE PERTURBATION APPROACH

A. Polarized Lidar Sounding with Regard to Multiple Scattering

Although not discussing lidar sounding in detail, we consider an application of lidar sounding to cloud remote sensing. The cloud as a scattering medium is characterized by large optical thickness and a highly extended phase function, which requires one to use different techniques to simulate lidar return rather than the single-scattering approximation, which is basic in lidar sounding theory. Monte Carlo[40,41] allows one to simulate the problem with high accuracy but provides poor ability to perform qualitative studies. Semianalytical techniques, such as those discussed in Refs. 29 and 42, close this gap by combining a clear physical model of the phenomenon with computational efficiency. We have chosen this complex problem to show how efficient the perturbation technique is in analyzing it and reducing it to the solution of a simpler problem.

Let us consider a typical problem of cloud-polarized sounding by a monostatic lidar, when a stratified scattering medium is illuminated along the normal by a δ pulse of linearly polarized light. The reflected light is detected by a receiver, which is placed at the same location $\mathbf{r}_0$ as the source and is characterized by its field of view. To calculate the signal, we have to solve Eq. (1). The common method for the solution of this problem is Monte Carlo,[40,41] and it demands a lot of computer time to perform even a single simulation, making qualitative analysis of the whole problem in such a situation a very awkward and expensive procedure. The perturbation technique provides us with a different approach. First, we should subdivide our problem into two parts: base and perturbation. To do so, a good starting point is to consider the cloud-scattering matrix. Its property of being highly extended, which is an obstacle during numerical simulation, is the one we are looking for. Because of this, the probability for a photon to be scattered into a direction that is close to the initial one is very high. Thus let the scattering matrix be represented as a sum of two parts:

$$\mathbf{P}(z, \cos\beta) = \mathbf{P}_b(z, \cos\beta) + \delta\mathbf{P}(z, \cos\beta). \quad (36)$$

Here $\mathbf{P}_b(z, \cos\beta)$ determines scattering over scattering angles $\beta < 90°$ and $\delta\mathbf{P}(z, \cos\beta)$ describes the backward hemisphere scattering ($\beta > 90°$).

We know[1,3] that $\mathbf{P}_b(z, \cos\beta)$ chiefly determines the characteristics of the radiation that propagates into directions that are close to the direction of illumination. We may introduce an effective medium that is characterized by the extinction coefficient $\sigma_e(z)$, scattering coefficient $\alpha\sigma_s(z)$, and phase matrix $\mathbf{P}_b(z, \cos\beta)/\alpha$, where $\alpha = \int_0^{\pi/2} a_{11}(\cos\beta)\sin\beta \mathrm{d}\beta$, and consider it as the base medium. (For water droplet cloud models,[43] numerical calculation shows that $\alpha \leqslant 0.95$.) In this case $\delta\mathbf{P}(z, \cos\beta)$ is the perturbation, and, with the help of approximation (33), we can immediately write the expression for the lidar signal $F(t)$ as

$$F(t) = F_b(t) + \left\langle \tilde{\mathbf{I}}_b^{+}(t, \mathbf{r}, \mathbf{n}), \frac{\sigma_s(z)}{4\pi}\int_{4\pi}\delta\mathbf{Z}(z, \mathbf{n}, \mathbf{n}') \times \mathbf{I}_b(t, \mathbf{r}, \mathbf{n}')\mathrm{d}\mathbf{n}' \right\rangle. \quad (37)$$

Here $F_b(t)$ is the lidar signal component reflected from the base medium and $\mathbf{I}_b(t, \mathbf{r}, \mathbf{n})$ is the solution of the equation

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{n}\cdot\nabla + \sigma_e(\mathbf{r})\right]\mathbf{I}_b(t, \mathbf{r}, \mathbf{n}) = \frac{\sigma_s^b(\mathbf{r})}{4\pi}\int_{4\pi}\mathbf{Z}_b(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{I}_b(t, \mathbf{r}, \mathbf{n}')\mathrm{d}\mathbf{n}' + \mathbf{Q}_0\delta(t - t_0)\delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{n} - \mathbf{e}_z), \quad (38)$$

where $\sigma_s^b = \alpha\sigma_s$ is the scattering coefficient of the base medium, $\mathbf{Q}_0$ is the Stokes vector of the source radiation, t_0 is the moment at which the source radiates the δ pulse, and $\mathbf{e}_z$ is the unit vector directed along the positive direction of the z axis. $\tilde{\mathbf{I}}_b(t, \mathbf{r}, \mathbf{n})$ is the solution of the corresponding adjoint problem.

At this moment the objection might be raised that we need to solve a problem more difficult than the initial one: In addition to the solution of Eq. (38), which is very similar to Eq. (1), we need to find the solution of the adjoint problem and perform an additional calculation to estimate the integral in Eq. (37). However, for the base medium we can state that

1. The single-scattering albedo ω_0^b of our base medium is determined by

$$\omega_0^b(z) = \alpha\frac{\sigma_s(z)}{\sigma_e(z)} \leqslant \alpha, \quad (39)$$

and, taking into account that for water droplet clouds $\alpha \leqslant 0.95$, we may consider our base medium as having considerable absorption.

2. The cloud phase function is highly extended.

Thus to solve Eq. (38) and the corresponding adjoint problem, we may use the small-angle approximation.[1,3] According to the small-angle approximation:

1. The lidar signal reflected from the base medium can be neglected; that is, $F_b(t) \approx 0$.

2. The temporal deformation of the propagating pulse can be neglected up to considerable optical depth, so the Stokes vector of the radiance has the form

$$\mathbf{I}_b(t, \mathbf{r}, \mathbf{n}) = \mathbf{I}_b(\mathbf{r}, \mathbf{n})\delta\left(t - t_0 - t_c - \frac{z}{c}\right), \quad (40)$$

where t_c is the time that is necessary for the pulse to travel between the source and the nearest cloud boundary and $\mathbf{I}_b(\mathbf{r}, \mathbf{n})$ is the solution of the stationary small-angle radiative transfer equation (more details can be found in Ref. 29):

$$[\mathbf{n}^s\cdot\nabla + \sigma_e(\mathbf{r})]\mathbf{I}_b(\mathbf{r}, \mathbf{n}) = \frac{\sigma_s(\mathbf{r})}{4\pi}\int_{4\pi}\mathbf{Z}_b^s(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{I}_b(\mathbf{r}, \mathbf{n}')\mathrm{d}\mathbf{n}' + \mathbf{Q}_0\delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{n} - \mathbf{e}_z). \quad (41)$$

Here $\mathbf{n}^s = (n_z = 1, n_x, n_y)$, $n_x \ll 1$, $n_y \ll 1$, and

$$\mathbf{Z}_b^s(z,\ \mathbf{n},\ \mathbf{n}') = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & \dfrac{a_2 + a_3}{2} & 0 & 0 \\ 0 & 0 & \dfrac{a_2 + a_3}{2} & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}, \quad (42)$$

where a_k are the same as in Eq. (5).

As mentioned above, for the lidar sounding problem the solutions of the direct and adjoint problems are similar, and all results obtained for the direct problem can be used for the adjoint. So we have managed to significantly simplify the initial problem: Instead of dealing with a system of four time-dependent equations [Eq. (1)], we need to solve only four independent stationary equations and then perform a simple integration to obtain the lidar signal. The quasi-analytical technique to solve Eq. (41) and its accuracy are discussed in Ref. 44.

The accuracy of such an approximation may be demonstrated by means of comparison with a Monte Carlo simulation. To do this, we calculate the depolarization degree

$$\delta_d = \frac{I - Q}{I + Q} \quad (43)$$

of the lidar signal reflected from a homogeneous cloud, which is 1000 m distance from the lidar system. The cloud optical properties are described by Deirmendjian's model Cloud C.1 at 1.064 μm[43] with an extinction coefficient of 0.02 m^{-1}. The beam divergence of the source is 0.0005 rad. Two values of the receiver field of view are considered: 0.0005 and 0.005 rad. The result of the comparison is given in Fig. 1. The Monte Carlo data were provided by Bruscaglioni:[45] The figure shows that the results coincide quite well, whereas the time necessary to make the calculation by use of this technique is considerably less than for the Monte Carlo simulation.

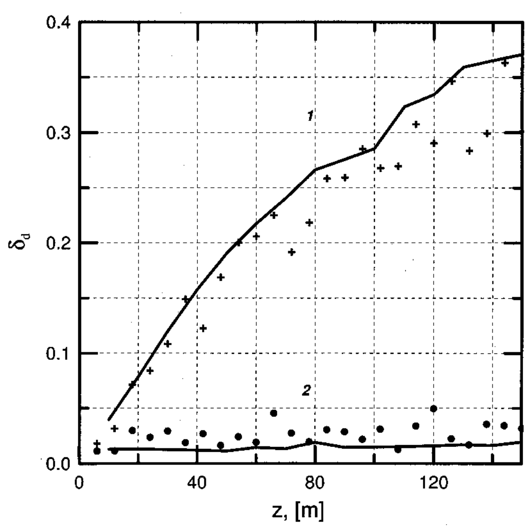

Fig. 1. Depolarization degree δ_d of the lidar signal reflected from a homogeneous cloud, as a function of the sounding depth z, computed by use of the perturbation approach (solid curve) and results of the Monte Carlo simulation (crosses and circles). The receiver field of view is (1) 0.005 and (2) 0.0005.

Note that the small oscillation of the curves that depict the perturbation technique estimation are a result of the simulation of I and Q separately and only then their difference, which clearly leads to increased relative errors. However, the amplitude of this oscillation is much smaller than the variability of the Monte Carlo simulation results.

The accuracy of numerical prediction, despite its importance, is a relatively minor advantage of this technique to model the lidar signal. The most powerful feature of this approach is its ability to provide a final result in semianalytical form that allows one to perform a qualitative study. For example, on substituting into Eq. (37) a solution of Eq. (41) obtained within the framework of the small-angle approximation, it may be deduced that the total power of the lidar signal for a satellite-based lidar has the form[46]

$$F(t) = (1 - \alpha) W_0 \Sigma_{\mathrm{rec}} \frac{\sigma_s^b(z) \exp[-2\tau(z)]}{\pi (H + z)^2} \times \int_0^\infty b(k) \exp[2\omega_0 \tau(z) p(k)] k \mathrm{d}k, \quad (44)$$

where W_0 is the pulse energy, Σ_{rec} is the receiver area, $\tau(z) = \int_0^z \sigma_e^b(z')\mathrm{d}z'$, $b(k) = \int_0^{\pi/2} a_1(\pi - \beta) J_0(k\beta)\beta \mathrm{d}\beta$, and $p(k) = \int_0^{\pi/2} a_1(\beta) J_0(k\beta)\beta \mathrm{d}\beta$. The form of Eq. (44) makes clear that only the extinction coefficient profile of water droplet clouds can be retrieved by use of satellite-based lidar as $\int_0^\infty b(k)\exp[2\omega_0\tau(z)p(k)]k\mathrm{d}k$ depends very weakly on the effective radius of cloud particles.

Despite the fact that our formulas are the same as those obtained by Zege and Chaikovskaya,[29] nevertheless the most important point we would like to make is that the perturbation approach does provide a simpler and more efficient way to obtain the basic formulas with a clear understanding of the errors introduced.

B. Three-Dimensional Radiative Transfer

One of the problems of passive cloud satellite remote sensing might be formulated as follows—how does the inhomogeneity of the optical properties of the cloud field affect the radiance characteristics? Two numerical approaches to obtain a solution may be noted: Monte Carlo[40] and the spherical harmonics discrete-ordinate method[47] (SHDOM). However, both of them are very time consuming, and they provide limited ability for qualitative study, especially if the effect of the cloud field structure is under investigation, when one has to recalculate the whole problem for each particular case. That is why such approaches do not look attractive, especially when there is no clear requirement for high accuracy, and an approximate solution is likely to be sufficient to estimate the contribution of the above effect.

Let us consider this problem from the point of view of the perturbation approach. It is clear that a medium, the optical characteristics of which are an average of the corresponding properties of the cloud field, should be chosen as the base case. The perturbation is then the difference between the characteristics of the base and real media. To be more specific, the corresponding parameters have the form

$$\sigma_e^b(z) = \frac{\int \sigma_e(\mathbf{r})\mathrm{d}x\mathrm{d}y}{\int \mathrm{d}x\mathrm{d}y}, \qquad \delta\sigma_e(\mathbf{r}) = \sigma_e(\mathbf{r}) - \sigma_e^b(z), \tag{45a}$$

$$\sigma_s^b(z) = \frac{\int \sigma_s(\mathbf{r})\mathrm{d}x\mathrm{d}y}{\int \mathrm{d}x\mathrm{d}y}, \qquad \delta\sigma_s(\mathbf{r}) = \sigma_s(\mathbf{r}) - \sigma_s^b(z), \tag{45b}$$

$$\mathbf{Z}^b(z,\ \mathbf{n},\ \mathbf{n}') = \frac{\int \sigma_s(\mathbf{r})\mathbf{Z}(\mathbf{r},\ \mathbf{n},\ \mathbf{n}')\mathrm{d}x\mathrm{d}y}{\int \sigma_s(\mathbf{r})\mathrm{d}x\mathrm{d}y},$$

$$\delta\mathbf{Z}(\mathbf{r},\ \mathbf{n},\ \mathbf{n}') = \frac{\sigma_s(\mathbf{r})}{\sigma_s^b(z)}[\mathbf{Z}(\mathbf{r},\ \mathbf{n},\ \mathbf{n}') - \mathbf{Z}^b(z,\ \mathbf{n},\ \mathbf{n}')], \tag{45c}$$

where the integration is presumed to be performed over the whole area of interest (for example, it is the whole image area for the satellite observation). Assuming solar illumination and a satellite sensor as the receiver, we may write the corresponding source and receiver functions in the form

$$\mathbf{Q}(t,\ \mathbf{r},\ \mathbf{n}) = \mathbf{Q}_0\delta(z - z_t)\delta(\mathbf{n} - \mathbf{n}_0), \tag{46a}$$

$$\mathbf{R}(t,\ \mathbf{r},\ \mathbf{n}) = \mathbf{R}_0\delta(\mathbf{r} - \mathbf{r}_r)\delta(\mathbf{n} - \mathbf{n}_r). \tag{46b}$$

Here $\mathbf{Q}_0$ and $\mathbf{R}_0$ are the Stokes vectors that characterize the source and receiver polarization states, respectively, z_t is the coordinate of the atmosphere top, $\mathbf{n}_0$ is the direction of solar illumination, $\mathbf{r}_r$ is the receiver location, and $\mathbf{n}_r$ is the direction of the receiver observation.

Equations (45) and (46) completely determine the symmetry properties of both the direct and the adjoint problems for the base medium. As our medium is horizontally homogeneous, the number of VRTE dimensions of either the direct or the adjoint problem is determined by the corresponding source function symmetry. In particular, we may conclude that to calculate the base case effect E_b and to obtain the direct solution $\mathbf{I}_b(z,\ \mathbf{n})$, we need to solve only the one-dimensional VRTE. However, to obtain the adjoint radiance $\tilde{\mathbf{I}}_b^+(\mathbf{r},\ \mathbf{n})$, the 3-D VRTE has to be solved but for a horizontally homogeneous medium with possible vertical stratification. This substantially simplifies the calculation problem.

From Eqs. (45) it is clear that the perturbation of the radiative transfer operator has the form

$$\delta\hat{\mathbf{L}} = \delta\sigma_e(\mathbf{r}) - \frac{1}{4\pi}\int_{4\pi}\mathrm{d}\mathbf{n}'[\delta\sigma_s(\mathbf{r})\mathbf{Z}^b(z,\ \mathbf{n},\ \mathbf{n}') + \sigma_s^b(z)\delta\mathbf{Z}(\mathbf{r},\ \mathbf{n},\ \mathbf{n}')]\circ. \tag{47}$$

Approximation (33) then gives us the answer we are looking for:

$$E = E_b - \langle\tilde{\mathbf{I}}_b^+(\mathbf{r},\ \mathbf{n}), \delta\hat{\mathbf{L}}\mathbf{I}_b(z,\ \mathbf{n}')\rangle. \tag{48}$$

Neglecting polarization and using the problem symmetry, we may rewrite this formula to describe the upwelling radiance variation as

$$\delta I(\mathbf{r}_{\perp r}) = -\iint\left[\int I_b^+(z,\ \mathbf{r}_\perp - \mathbf{r}_{\perp r},\ \mathbf{n})\delta\hat{\mathbf{L}}(z,\ \mathbf{r}_\perp,\ \mathbf{n})\mathrm{d}\mathbf{r}_\perp\right] \times I_b(z,\ \mathbf{n})\mathrm{d}z\mathrm{d}\mathbf{n}, \tag{49}$$

where $\mathbf{r}_\perp$ is the projection of the vector $\mathbf{r}$ on the xy plane of our coordinate system. Note that in cases in which $\delta\hat{\mathbf{L}}$ does not depend on $\mathbf{r}_\perp$ Eq. (49) becomes simpler as $I_b^+(z,\ \mathbf{n}) = \int I_b^+(z,\ \mathbf{r}_\perp - \mathbf{r}_{\perp r},\ \mathbf{n})$, the necessity of a 3-D solution of the adjoint problem vanishes, and we come to an equation similar to the one obtained in Ref. 25.

We now underline some advantages of the above solution.

1. The accuracy of Eq. (48) is high enough if the variation of the optical properties is not large even in the case of an optically thick medium because the single-scattering albedo $\omega_0 \approx 1$ $(1 - \omega_0 \ll 1)$ and asymmetry parameter $g \approx 0.85$ can be considered constants[35] for a water droplet cloud in the visible. The last statement is important to prevent the occurrence of substantial differences between the radiances in the base and perturbed media.
2. It is not necessary to recalculate the whole problem from the very beginning to estimate the effect of different perturbations but only to recalculate the integral of Eq. (48) for different forms of $\delta\hat{\mathbf{L}}$.

The accuracy of the perturbation approach may be demonstrated by solution of one of the simplest problems of satellite observation of a cloud field with a sine-type modulation of the extinction coefficient. This test problem has been considered previously.[8,9,48]

We consider a slab of a scattering medium that is characterized by the single-scattering albedo $\omega_0 = 0.999$ and a Henyey–Greenstein phase function,[49] defined as

$$P(\cos\beta) = \frac{1 - g^2}{[1 + g^2 - 2g\cos(\beta)]^{3/2}}, \tag{50}$$

where β is the scattering angle and g is the asymmetry parameter. The base medium is a homogeneous slab characterized by the extinction coefficient σ_e^b, which was chosen to provide an average optical thickness of 10.0. The extinction coefficient variation was chosen in the form

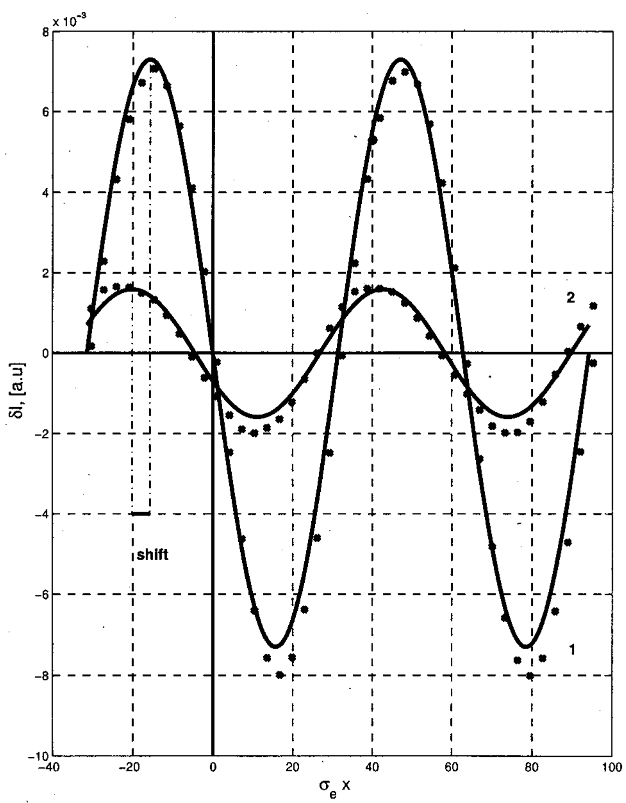

Fig. 2. Variation $\delta I(x)$ of the upwelling radiation (arbitrary units) at the top of the cloud as a function of $\sigma_e^b x$ calculated by use of the perturbation approach (solid curve) and the SHDOM (crosses). The cosine μ_0 of the solar zenith angle is (1) 1.0 and (2) 0.3.

$$\delta\sigma_e(\mathbf{r}) = \epsilon\sigma_e^b \sin(k_0\sigma_e^b x), \tag{51}$$

with $k_0 = 0.1$ and $\epsilon = 0.1$. The plane of incidence of the solar beam coincides with the xz plane, and the receiver measures the upwelling radiance along the normal to the slab boundary.

To validate the accuracy of the perturbation calculation, we also simulated this problem employing the SHDOM. For the SHDOM calculation a grid of 120-by-50 cells and periodic horizontal boundary conditions were used. The angular resolution was 16 streams with 32 azimuthal modes. The terminology is borrowed from Evans's paper,[47] in which all details about the SHDOM can be found. Figure 2 contains the results of the calculation of $\delta I(x)$ with respect to $\sigma_e^b x$ for $g = 0.5$ by use of Eq. (49) (solid curves) and the SHDOM (crosses). The solar zenith angle's cosines of $\mu_0 = 1.0$ (curve 1) and $\mu_0 = 0.3$ (curve 2) are considered. We can see that the phase of the oscillation of the observed upwelling radiation along the x axis for $\mu_0 = 0.3$ does not coincide with the phase of the extinction coefficient modulation, which is the same as for $\mu_0 = 1.0$. We call this phenomenon the shift, following Li *et al.*[8] As seen from the figure, the perturbation technique can describe the magnitude of both the oscillation and the shift, which appears in the case of oblique illumination. However, we should note that there are some minor deviations of the perturbation estimation from the SHDOM results. The main reason is that the perturbation approach takes into account only the first-order effect of $\delta\hat{\mathbf{L}}$, and, to describe these deviations, the next-order effects should be taken into account, at least the second order as was done by Li *et al.*[9]

6. CONCLUSION

In this paper we have considered the general formulation of the perturbation technique relevant to the optics of scattering media. We also demonstrated through two examples the main advantage of the perturbation technique, which is undoubtedly that it provides an efficient and simple way to study complex problems, whose solution otherwise demands large amounts of computer time. Certainly the advantage of the perturbation approach tends to be more prominent the greater the number of different perturbations that are under consideration.

For some applications the perturbation technique is an approximation. This situation includes problems that assume estimation of how given radiance characteristics respond to medium perturbation of a given magnitude and requires the use of approximation (33). In these circumstances the accuracy cannot be easily estimated. Fortunately, there are many problems the formulation of which guarantees the smallness of the medium perturbation, such as problems of target detection and identification, when the perturbation of the radiance field generated by the target must be assumed to be small (otherwise, there is no problem in detecting the target at all). For other problems, the perturbation series expansion[14] may be considered, and the error can be estimated on the basis of calculation of the second term of the series.

More favorable applications are those that require a study of sensitivity, as the perturbation technique provides a simple and accurate recipe for the corresponding derivative calculation. In this situation the accuracy as follows from Eq. (35) is fully determined by the quality of the solution of the unperturbed radiative transfer equation.

Unfortunately, we cannot consider all possible applications of the perturbation technique to scattering medium optics; however, we have attempted to give two quite different examples. We will devote specific papers to consider some of these problems in more detail in the future.

APPENDIX A: DERIVATION OF THE BOUNDARY CONDITIONS FOR THE ADJOINT PROBLEM

Substituting Eqs. (12) and (17) into Eq. (15) and taking into account the definition of the scalar product [Eq. (14)], we obtain

$$\langle \tilde{\mathbf{I}}^+, \hat{\mathbf{L}}\mathbf{I}\rangle - \langle(\hat{\tilde{\mathbf{L}}}\tilde{\mathbf{I}}^+)^+, \mathbf{I}\rangle$$

$$= \int\int\int_\Xi \left\{\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\frac{1}{c}\frac{\partial}{\partial t}\mathbf{I}(t, \mathbf{r}, \mathbf{n}) + \left[\frac{1}{c}\frac{\partial}{\partial t}\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\right]\mathbf{I}(t, \mathbf{r}, \mathbf{n})\mathrm{d}\Xi\right.$$
$$+ \int\int\int_\Xi \{\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{n}\cdot\nabla\mathbf{I}(t, \mathbf{r}, \mathbf{n}) + [\mathbf{n}\cdot\nabla\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})]\mathbf{I}(t, \mathbf{r}, \mathbf{n})\}\mathrm{d}\Xi$$
$$+ \int\int\int_\Xi \frac{\sigma_s(\mathbf{r})}{4\pi}\int_{4\pi}[\mathbf{TZ}(\mathbf{r}, -\mathbf{n}, -\mathbf{n}') \times \mathbf{TI}(t, \mathbf{r}, \mathbf{n}')]^+\mathbf{I}(t, \mathbf{r}, \mathbf{n})\mathrm{d}\mathbf{n}'\mathrm{d}\Xi$$
$$- \int\int\int_\Xi \tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\frac{\sigma_s(\mathbf{r})}{4\pi} \times \int_{4\pi}\mathbf{Z}(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{I}(t, \mathbf{r}, \mathbf{n}')\mathrm{d}\mathbf{n}'\mathrm{d}\Xi. \quad \text{(A1)}$$

Applying the reciprocity relationship [Eq. (3)] to the phase matrix makes the last two terms in the right-hand side of Eq. (A1) vanish because

$$\int\int\int_\Xi \tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\sigma_s\int_{4\pi}\mathbf{Z}(\mathbf{r}, \mathbf{n}, \mathbf{n}')\mathbf{I}(t, \mathbf{r}, \mathbf{n}')\mathrm{d}\mathbf{n}'\mathrm{d}\Xi$$
$$= \int\int\int_\Xi\int_{4\pi}\sigma_s[\mathbf{Z}^+(\mathbf{r}, \mathbf{n}', \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n}')]^+\mathbf{I}(t, \mathbf{r}, \mathbf{n})\mathrm{d}\mathbf{n}'\mathrm{d}\Xi$$
$$= \int\int\int_\Xi\int_{4\pi}\sigma_s[\mathbf{TZ}(\mathbf{r}, -\mathbf{n}, -\mathbf{n}')\mathbf{TI}(t, \mathbf{r}, \mathbf{n}')]^+ \times \mathbf{I}(t, \mathbf{r}, \mathbf{n})\mathrm{d}\mathbf{n}'\mathrm{d}\Xi. \quad \text{(A2)}$$

Taking into account Eq. (7), it is clear that

$$\int\int\int_\Xi \left\{\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\frac{1}{c}\frac{\partial}{\partial t}\mathbf{I}(t, \mathbf{r}, \mathbf{n}) + \left[\frac{1}{c}\frac{\partial}{\partial t}\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\right]\mathbf{I}(t, \mathbf{r}, \mathbf{n})\mathrm{d}\Xi\right.$$
$$= \int\int\int_\Xi \frac{1}{c}\frac{\partial}{\partial t}[\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})]\mathrm{d}\Xi$$
$$= \frac{1}{c}\int\int [\tilde{\mathbf{I}}^+(t_2, \mathbf{r}, \mathbf{n})\mathbf{I}(t_2, \mathbf{r}, \mathbf{n}) - \tilde{\mathbf{I}}^+(t_1, \mathbf{r}, \mathbf{n})\mathbf{I}(t_1, \mathbf{r}, \mathbf{n})]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{n}, \quad \text{(A3)}$$

where $t_2 > t_1$. Applying the divergence theorem in the form

$$\int_W \mathbf{n}\cdot\nabla[\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})]\mathrm{d}\mathbf{r} = \int_S \tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})(\mathbf{n}\cdot\mathbf{n}_\perp)\mathrm{d}S, \quad \text{(A4)}$$

where W means the integration over the geometrical volume of the medium and $\mathbf{n}_\perp$ is the normal to the boundary surface, we obtain

$$\int\int\int_\Xi \{\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{n}\cdot\nabla\mathbf{I}(t, \mathbf{r}, \mathbf{n}) + [\mathbf{n}\cdot\nabla\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})]\mathbf{I}(t, \mathbf{r}, \mathbf{n})\}\mathrm{d}\Xi$$
$$= \int\int\int_\Xi \mathbf{n}\cdot\nabla[\tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})]\mathrm{d}\Xi$$
$$= \int\int \left[\int_S \tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})(\mathbf{n}\cdot\mathbf{n}_\perp)\mathrm{d}S\right]\mathrm{d}t\mathrm{d}\mathbf{n}. \quad \text{(A5)}$$

To satisfy Eq. (15) as it follows from Eq. (A1), it is necessary that the following equation,

$$\frac{1}{c}\int\int [\tilde{\mathbf{I}}^+(t_2, \mathbf{r}, \mathbf{n})\mathbf{I}(t_2, \mathbf{r}, \mathbf{n}) - \tilde{\mathbf{I}}^+(t_1, \mathbf{r}, \mathbf{n})\mathbf{I}(t_1, \mathbf{r}, \mathbf{n})]\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{n} + \int\int \left[\int_S \tilde{\mathbf{I}}^+(t, \mathbf{r}, \mathbf{n})\mathbf{I}(t, \mathbf{r}, \mathbf{n})(\mathbf{n}\cdot\mathbf{n}_\perp)\mathrm{d}S\right]\mathrm{d}t\mathrm{d}\mathbf{n} = 0, \quad \text{(A6)}$$

must be fulfilled. This is Eq. (18).

APPENDIX B: DERIVATION OF EQ. (30)

The derivation of Eq. (30) is similar to the one used by Box *et al.*[15] and is based on the identities

$$\hat{\mathbf{L}}_p\mathbf{I}_p = \hat{\mathbf{L}}_b\mathbf{I}_b = \mathbf{Q}, \quad \text{(B1a)}$$

$$\hat{\tilde{\mathbf{L}}}_p\tilde{\mathbf{I}}_p = \hat{\tilde{\mathbf{L}}}_b\tilde{\mathbf{I}}_b = \mathbf{R}. \quad \text{(B1b)}$$

The variation δE of the effect E may be represented as

$$\delta E = \langle\mathbf{R}^+, \mathbf{I}_p\rangle - \langle\mathbf{R}^+, \mathbf{I}_b\rangle = \langle\mathbf{R}^+, \delta\mathbf{I}\rangle = \langle(\hat{\tilde{\mathbf{L}}}_b\tilde{\mathbf{I}}_b)^+, \delta\mathbf{I}\rangle = \langle\tilde{\mathbf{I}}_b^+, \hat{\mathbf{L}}_b\delta\mathbf{I}\rangle, \quad \text{(B2)}$$

where $\delta\mathbf{I} = \mathbf{I}_p - \mathbf{I}_b$.

Let us consider the identity

$$\hat{\mathbf{L}}_p\mathbf{I}_p - \hat{\mathbf{L}}_b\mathbf{I}_p = \hat{\mathbf{L}}_b\mathbf{I}_b - \hat{\mathbf{L}}_b\mathbf{I}_p, \quad \text{(B3)}$$

which may be rewritten as

$$\delta\hat{\mathbf{L}}\mathbf{I}_p = -\hat{\mathbf{L}}_b\delta\mathbf{I}. \quad \text{(B4)}$$

After substitution of Eq. (B4) into Eq. (B2), we may perform further simple manipulations:

$$\begin{aligned}\delta E &= -\langle \tilde{\mathbf{I}}_b^+,\ \delta\tilde{\mathbf{L}}\mathbf{I}_p\rangle = -\langle \tilde{\mathbf{I}}_b^+,\ \delta\tilde{\mathbf{L}}\mathbf{I}_b\rangle - \langle \tilde{\mathbf{I}}_b^+,\ \delta\hat{\mathbf{L}}\delta\mathbf{I}\rangle \\ &= -\langle \tilde{\mathbf{I}}_b^+,\ \delta\hat{\mathbf{L}}\mathbf{I}_b\rangle - \langle(\delta\hat{\tilde{\mathbf{L}}}\mathbf{I}_b)^+,\ \delta\mathbf{I}\rangle \\ &= -\langle \tilde{\mathbf{I}}_b^+,\ \delta\hat{\mathbf{L}}\mathbf{I}_b\rangle - \langle(\hat{\tilde{\mathbf{L}}}_p\mathbf{I}_b)^+,\ \delta\mathbf{I}\rangle + \langle(\hat{\tilde{\mathbf{L}}}_b\tilde{\mathbf{I}}_b)^+,\ \delta\mathbf{I}\rangle \\ &= -\langle \tilde{\mathbf{I}}_b^+,\ \delta\hat{\mathbf{L}}\mathbf{I}_b\rangle + \langle(\hat{\tilde{\mathbf{L}}}_p\delta\tilde{\mathbf{I}})^+,\ \delta\mathbf{I}\rangle - \langle(\hat{\tilde{\mathbf{L}}}_p\tilde{\mathbf{I}}_p)^+,\ \delta\mathbf{I}\rangle \\ &\quad + \langle(\hat{\tilde{\mathbf{L}}}_b\tilde{\mathbf{I}}_b)^+,\ \delta\mathbf{I}\rangle. \end{aligned} \tag{B5}$$

Finally, taking into account Eq. (B1), we obtain the basic formula of the perturbation approach:

$$\delta E = -\langle \tilde{\mathbf{I}}_b^+,\ \hat{\mathbf{L}}\mathbf{I}_b\rangle + \langle \delta\tilde{\mathbf{I}}^+,\ \hat{\mathbf{L}}_p\delta\mathbf{I}\rangle. \tag{B6}$$

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The corresponding author, I. N. Polonsky, can be reached at the address on the title page or by e-mail, polonsky@phys.unsw.edu.au.

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