Journal of Quantitative Spectroscopy & Radiative Transfer 84 (2004) 105–114

Journal of Quantitative Spectroscopy & Radiative Transfer

www.elsevier.com/locate/jqsrt

Higher-order radiative perturbation theory

Yi Qin[a], Michael A. Box[a,*], Thomas Trautmann[b]

[a] School of Physics, University of New South Wales, Sydney, NSW 2052, Australia

[b] Institute of Meteorology, University of Leipzig, Germany

Received 4 November 2002; accepted 11 March 2003

Abstract

As a computationally effective tool, the first-order term of the radiative perturbation theory has been computed successfully, and has been applied in a number of areas. In this article, we develop the computational expressions for the higher-order terms of the perturbation expansion in a plane parallel atmosphere. These expressions are then implemented, and numerical results for some typical cases are presented. These results indicate that the computation is successful and that the higher-order terms are essential in cases where the first-order term alone cannot predict the perturbation with sufficient accuracy.

Keywords: Radiative transfer; Green's function; Higher-order perturbation

1. Introduction

The computation of first-order radiative perturbation theory has been quite successful, and has been applied to problems in aerosol atmospheres [1,2], spectral integration [3,4] and atmospheric property remote sensing [5–7]. A review has been given by Box [8]. The theory of higher-order perturbation (HOP) has been discussed by Box et al. [9] and attempts have been made to obtain the Green's function [10] and numerical results for higher-order terms for some special cases [11]. However, no numerical results are yet available for general atmospheric models: for example, vertically variable and anisotropically scattering atmospheric models. In this article, we will develop the computational expressions of the HOP and present some numerical results for general atmospheric models.

The first-order perturbation (FOP) has been found to account for the most significant part of radiative effect perturbation. Obtaining this part as precisely as possible will be critically important for

* Corresponding author. Tel.: +61-2-9385-4545; fax: +61-2-9385-6060.
E-mail address: m.box@unsw.edu.au (M.A. Box).

0022-4073/$ - see front matter
doi:10.1016/S0022-4073(03)00138-9

HOP computation. Also because its computation is quite important, a separate paper will be dedicated to it (manuscript in preparation). In the present paper, we will concentrate on the higher-order terms.

2. The computational equations of HOP

It has been shown by Box et al. [9] that a radiative effect (such as flux or heating rate) of a given atmosphere model can be generally expressed in terms of a base model value as

$$E = E_0 - \langle I_0^+, \Delta L I_0 \rangle + \langle I_0^+, \Delta L G_0 \Delta L I_0 \rangle - \langle I_0^+, \Delta L G_0 \Delta L G_0 \Delta L I_0 \rangle + \cdots, \tag{1}$$

where angular brackets denote integration over all relevant space and angle coordinates, $(x, y, z; \mu, \phi)$, which constitute the phase space of the problem.

In this equation, E is the radiative effect of concern for a given atmospheric model, and E_0 is the same effect corresponding to a base model. I_0 is the forward radiance function of the base model and I_0^+ is the adjoint radiance function corresponding to E_0 [8,9]. G_0 is the Green's function of the base model [9]. ΔL is the perturbation of the radiative transfer operator, which represents the perturbation of the atmospheric model [8,12] and will be explained in more detail later. Note also that in the above equation, G_0 acts as an operator, i.e., for any function $X(x', y', z', \mu', \phi')$,

$$G_0 X \equiv \langle G_0, X \rangle = \int \mathrm{d}x' \int \mathrm{d}y' \int \mathrm{d}z' \int_{-1}^{1} \mathrm{d}\mu' \int_0^{2\pi} \mathrm{d}\phi' \, G_0(x, y, z, \mu, \phi; x', y', z', \mu', \phi') X(x', y', z', \mu', \phi'). \tag{2}$$

Eq. (1) shows that any radiative effect for any given atmospheric model can be potentially obtained from the base model quantities without having to solve the radiative transfer equation for the given model and effect. However, it is still too complicated to consider general effects of general atmospheric models, such as the radiance of three-dimensional [13] or spherical atmospheric models. In this paper, we will consider the effect to be azimuth-independent (for example, fluxes and heating rate), and the atmosphere is assumed to be plane parallel.

2.1. Second-order term

Since the first-order term is covered in a separate paper, we start from the second order here, i.e.

$$\Delta_2 E \equiv \langle I_0^+, \Delta L G_0 \Delta L I_0 \rangle = \langle G_0^+ \Delta L^+ I_0^+, \Delta L I_0 \rangle, \tag{3}$$

where G_0^+ is the adjoint Green's function [9]. Because of the assumption of a plane parallel atmospheric model, no quantity in the above equation depends on x or y, so that integrations over (x, y, z) are reduced to integrations over z only.

Since we are considering azimuth-independent effects, the response function and the adjoint radiance, I_0^+, are azimuth-independent. We denote the adjoint radiance in this case as $\bar{I}_0^+(z, \mu)$. We recall that [8,12]

$$\Delta L = \Delta L^{+} = \int_{-1}^{1} \mathrm{d}\mu' \int_{0}^{2\pi} \mathrm{d}\phi' \left\{ \Delta\sigma_t(z)\delta(\mu - \mu')\delta(\phi - \phi') - \frac{1}{4\pi} \sum_{l=0}^{2N_i-1} (2l+1)\Delta\eta_l(z) \sum_{m=0}^{l} (2 - \delta_{0,m}) \frac{(l-m)!}{(l+m)!} P_l^m(\mu) P_l^m(\mu') \cos m(\phi - \phi') \right\} \circ \quad (4)$$

where $\Delta\sigma_{\mathrm{t}} = \sigma_{\mathrm{t}}^{(\mathrm{pert})} - \sigma_{\mathrm{t}}^{(\mathrm{base})}$ is the perturbation of the extinction cross section and $\Delta\eta_l = \sigma_{\mathrm{s}}^{(\mathrm{pert})}\chi_l^{(\mathrm{pert})} - \sigma_{\mathrm{s}}^{(\mathrm{base})}\chi_l^{(\mathrm{base})}$ represent the scattering perturbation, where σ_{s} is the scattering cross section and χ_l are the coefficients of the Legendre polynomial expansion of the base model or perturbed model phase function. N_i is related to the order of the expansion. Here the symbol $\circ$ is used as a place holder for the operand to which the operator applies. We see that integration over azimuth of $\Delta L^{+}\bar{I}_0^{+}$ can be readily carried out, giving

$$\Delta L^{+}\bar{I}_0^{+} \equiv H^{+}(z,\mu) = \Delta\sigma_{\mathrm{t}}(z)\bar{I}_0^{+}(z,\mu) - \sum_{l=0}^{2N_i-1} \Delta\eta_l(z) P_l(\mu)\xi_l^{+}(z) = \sum_{l=0}^{2N_i-1} \Delta Y_l(z) P_l(\mu)\xi_l^{+}(z), \quad (5)$$

where $\Delta Y_l(z) = \Delta\sigma_t(z) - \Delta\eta_l(z)$ denotes the model perturbation, and

$$\xi_l^{+}(z) = \frac{2l+1}{2} \int_{-1}^{1} P_l(\mu)\bar{I}_0^{+}(z,\mu)\,\mathrm{d}\mu \quad (6)$$

which is the coefficient when $\bar{I}_0^{+}(z,\mu)$ is expanded into a series of Legendre polynomials. Similarly, integration over azimuth of $G_0^{+}\Delta L^{+}\bar{I}_0^{+}$ can also be carried out, giving

$$G_0^{+}\Delta L^{+} I_0^{+} = \int \mathrm{d}z' \int_{-1}^{1} \mathrm{d}\mu' \int_{0}^{2\pi} \mathrm{d}\phi'\, G_0^{+}(z,\mu,\phi;z',\mu',\phi') H^{+}(z',\mu') = \int \mathrm{d}z' \int_{-1}^{1} \mathrm{d}\mu' \hat{G}_0^{+}(z,\mu;z',\mu') H^{+}(z',\mu'), \quad (7)$$

where $\hat{G}_0^{+}$ is the integration, or 2π times the average, of G_0^{+} over ϕ', which removes the dependence on ϕ as well. We see that $G_0^{+}\Delta L^{+} I_0^{+}$ depends on z and μ only, so that the outermost integration over azimuth in Eq. (3) applies to $\Delta L I_0$ only. We may write

$$\int_{0}^{2\pi} \mathrm{d}\phi\, \Delta L I_0 \equiv 2\pi H(z,\mu), \quad (8)$$

where

$$H(z,\mu) = \sum_{l=0}^{2N_i-1} \Delta Y_l(z) P_l(\mu)\xi_l(z) \quad (9)$$

and

$$\xi_l(z) = \frac{2l+1}{2} \int_{-1}^{1} P_l(\mu)\bar{I}_0(z,\mu)\,\mathrm{d}\mu, \tag{10}$$

which is the coefficient when $\bar{I}_0(z,\mu)$ is expanded into a series of Legendre polynomials. We now put all the elements together:

$$\Delta_2 E = 2\pi \int \mathrm{d}z \int_{-1}^{1} \mathrm{d}\mu \int \mathrm{d}z' \int_{-1}^{1} \mathrm{d}\mu'\, H(z,\mu)\hat{G}_0^+(z,\mu;z',\mu')H^+(z',\mu'). \tag{11}$$

2.2. Higher-order terms

Similarly for the third order, we have

$$\Delta_3 E \equiv \langle I_0^+, \Delta L G_0 \Delta L G_0 \Delta L I_0 \rangle = \langle G_0^+ \Delta L^+ G_0^+ \Delta L^+ I_0^+, \Delta L I_0 \rangle. \tag{12}$$

Because $G_0^+ \Delta L^+ I_0^+$ depends on z and μ only, integrations over azimuth can be carried out similarly:

$$\Delta_3 E = 2\pi \int \mathrm{d}z \int_{-1}^{1} \mathrm{d}\mu\, H(z,\mu) \int \mathrm{d}z_1 \int_{-1}^{1} \mathrm{d}\mu_1\, \hat{G}_0^+(z,\mu;z_1,\mu_1) \int_{-1}^{1} \mathrm{d}\mu_1'\, J(z_1,\mu_1;\mu_1')$$
$$\times \int \mathrm{d}z' \int_{-1}^{1} \mathrm{d}\mu' \hat{G}_0^+(z_1,\mu_1';z',\mu')H^+(z',\mu'), \tag{13}$$

where

$$J(z,\mu;\mu') = \Delta\sigma_\mathrm{t}(z)\delta(\mu'-\mu) - \sum_{l=0}^{2N_i-1} \frac{2l+1}{2}\,\Delta\eta_l(z)P_l(\mu')P_l(\mu)$$
$$= \sum_{l=0}^{2N_i-1} \frac{2l+1}{2}\,\Delta Y_l(z)P_l(\mu')P_l(\mu). \tag{14}$$

We may now write for the $(k+2)$th order $(k \geqslant 2)$:

$$\Delta_{k+2} E = 2\pi \int \mathrm{d}z \int_{-1}^{1} \mathrm{d}\mu\, H(z,\mu)$$
$$\times \int \mathrm{d}z_1 \int_{-1}^{1} \mathrm{d}\mu_1\, \hat{G}_0^+(z,\mu;z_1,\mu_1) \int_{-1}^{1} \mathrm{d}\mu_1'\, J(z_1,\mu_1;\mu_1')$$
$$\vdots$$
$$\times \int \mathrm{d}z_k \int_{-1}^{1} \mathrm{d}\mu_k\, \hat{G}_0^+(z_{k-1},\mu_{k-1};z_k,\mu_k) \int_{-1}^{1} \mathrm{d}\mu_k'\, J(z_k,\mu_k;\mu_k')$$
$$\times \int \mathrm{d}z' \int_{-1}^{1} \mathrm{d}\mu'\, \hat{G}_0^+(z_k,\mu_k';z',\mu')H^+(z',\mu'). \tag{15}$$

2.3. Green's function expansion

To further simplify the above equations, we expand $\hat{G}_0^+(z,\mu;z',\mu')$ as a double series of Legendre polynomials, i.e.,

$$\hat{G}_0^+(z,\mu;z',\mu') = \sum_m \sum_n \hat{g}_{mn}^+(z,z')P_m(\mu)P_n(\mu'), \tag{16}$$

the coefficients of which may be obtained from

$$\hat{g}_{mn}^+(z,z') = \frac{(2m+1)(2n+1)}{4} \int_{-1}^{1} P_m(\mu)\,\mathrm{d}\mu \int_{-1}^{1} \hat{G}_0^+(z,\mu,z',\mu')P_n(\mu')\,\mathrm{d}\mu'. \tag{17}$$

However, in an accompany paper [14], we show how to obtain them more efficiently in the discrete ordinate method formalism [15–20].

Inserting Eq. (16) into Eq. (11), and noting the orthogonality of Legendre polynomials, we obtain

$$\Delta_2 E = 2\pi \int \mathrm{d}z \sum_m \frac{2}{2m+1}\, \xi_m(z)\Delta Y_m(z) \int \mathrm{d}z' \sum_n \frac{2}{2n+1}\, \hat{g}_{mn}^+(z,z')\xi_n^+(z')\Delta Y_n(z'). \tag{18}$$

Similarly, we can find the expressions for the third and higher orders, which are summarized below

$$\begin{aligned}\Delta_3 E = 2\pi \int \mathrm{d}z \sum_m \frac{2}{2m+1} \xi_m(z)\Delta Y_m(z) \int \mathrm{d}z_1 \sum_{i_1} \frac{2}{2i_1+1}\, \hat{g}_{mi_1}^+(z,z_1)\Delta Y_{i_1}(z_1) \\ \times \int \mathrm{d}z' \sum_n \frac{2}{2n+1}\, g_{i_1 n}(z_1,z')\xi_n^+(z')\Delta Y_n(z'),\end{aligned} \tag{19}$$

$$\begin{aligned}\Delta_{k+2} E = 2\pi \int \mathrm{d}z \sum_m \frac{2}{2m+1}\, \xi_m(z)\Delta Y_m(z) \\ \times \int \mathrm{d}z_1 \sum_{i_1} \frac{2}{2i_1+1}\, \hat{g}_{mi_1}^+(z,z_1)\Delta Y_{i_1}(z_1) \\ \vdots \\ \times \int \mathrm{d}z_k \sum_{i_k} \frac{2}{2i_k+1}\, \hat{g}_{i_{k-1}i_k}^+(z_{k-1},z_k)\Delta Y_{i_k}(z_k) \\ \times \int \mathrm{d}z' \sum_n \frac{2}{2n+1}\, \hat{g}_{i_k n}^+(z_k,z')\xi_n^+(z')\Delta Y_n(z'),\end{aligned} \tag{20}$$

where $k \geqslant 2$. These equations show that higher-order terms can be obtained from lower-order terms by an iterative procedure, i.e.,

$$A_m^{(2)}(z) = \int \mathrm{d}z' \sum_n \frac{2}{2n+1}\, \hat{g}_{mn}^+(z,z')\xi_n^+(z')\Delta Y_n(z'), \tag{21}$$

$$A_m^{(k)}(z) = \int \mathrm{d}z' \sum_n \frac{2}{2n+1} A_n^{(k-1)}(z')\hat{g}_{mn}^{+}(z,z')\Delta Y_n(z'), \tag{22}$$

$$\Delta_k E = 2\pi \int \mathrm{d}z \sum_m \frac{2}{2m+1} A_m^{(k)}(z)\xi_m(z)\Delta Y_m(z). \tag{23}$$

3. Numerical computation

We assume that the atmosphere consists of N_z homogeneous layers, the optical properties of which may change from layer to layer. Because the total and scattering optical thicknesses are commonly used in defining atmospheric models, we need to convert them to cross-sections. Because of the assumption that each layer is homogeneous, for layer i, we have

$$\begin{aligned} \tau^{(i)} &= \sigma_\mathrm{t}^{(i)}\Delta z^{(i)}, \\ \tau_\mathrm{s}^{(i)} &= \sigma_\mathrm{s}^{(i)}\Delta z^{(i)}, \\ \tilde{\omega}_0^{(i)} &= \tau_\mathrm{s}^{(i)}/\tau^{(i)} = \sigma_\mathrm{s}^{(i)}/\sigma_\mathrm{t}^{(i)}, \end{aligned} \tag{24}$$

where $\tau^{(i)}$ and $\tau_\mathrm{s}^{(i)}$ are the total and scattering optical thicknesses of layer i. $\Delta z^{(i)}$ is the geometric thickness of layer i, which can be assumed to be any value without affecting the final numerical results. Although we have assumed that the atmosphere consists of a series of homogeneous layers, the radiance and Green's function still depend on altitude within each layer. To accomplish integrations over altitude, we evenly divide each layer, for example, layer i, $i = 1, 2, \ldots, N_z$ into $M_i - M_{i-1}$ sub-layers. The integration of any function $f(z)$ over z can then be computed using the trapezoidal rule:

$$\int_0^z f(z)\,\mathrm{d}z = \sum_{i=1}^{N_z} \int_{z_{i-1}}^{z_i} f(z)\,\mathrm{d}z = \sum_{i=1}^{N_z} \Delta z_i \sum_{j=M_{i-1}}^{M_j} a_j f(z_j), \tag{25}$$

where $a_j = 0.5$ if $j = M_{i-1}$ or M_i, otherwise $a_j = 1.0$, and $\Delta z_i = \Delta z^{(i)}/(M_i - M_{i-1})$ is the thickness of sub-layers within layer i.

We now conduct the computation outlined in Eqs. (21)–(23) for a few typical cases and present the results. Three cases are considered as shown in Table 1. All models consist of one homogeneous layer. The model phase function is represented by the two-term Henyey–Greenstein (TTHG) function defined as [21]:

$$P(\alpha, g_1, g_2, \Theta) = \alpha \frac{1 - g_1^2}{1 + g_1^2 - 2g_1\cos(\Theta)} + (1-\alpha)\frac{1 - g_2^2}{1 + g_2^2 + 2g_2\cos(\Theta)}, \tag{26}$$

where Θ is the scattering angle. This function is a linear combination of two Henyey–Greenstein functions. Each has a single parameter, g_1 or g_2, which is the asymmetry factor. Also shown in this table are the number of sub-layers and the number of streams used in the calculation.

Table 1
The cases considered for numerical computation

Case no.	Base model		Perturbed model		Number of sub-layers	Number of streams
	τ	$\tilde{\omega}_0$	τ	$\tilde{\omega}_0$		
1	1.00	1.00	2.0	0.975	50	32
2	10.0	1.00	15.0	1.00	250	8
3	10.0	1.00	10.0	0.975	250	8
Other parameters						
Phase function (all models)			TTHG (0.965, 0.75, 0.65)			
Surface albedo			0.0			
Solar Zenith angle			45°			
Solar radiance			1.0			

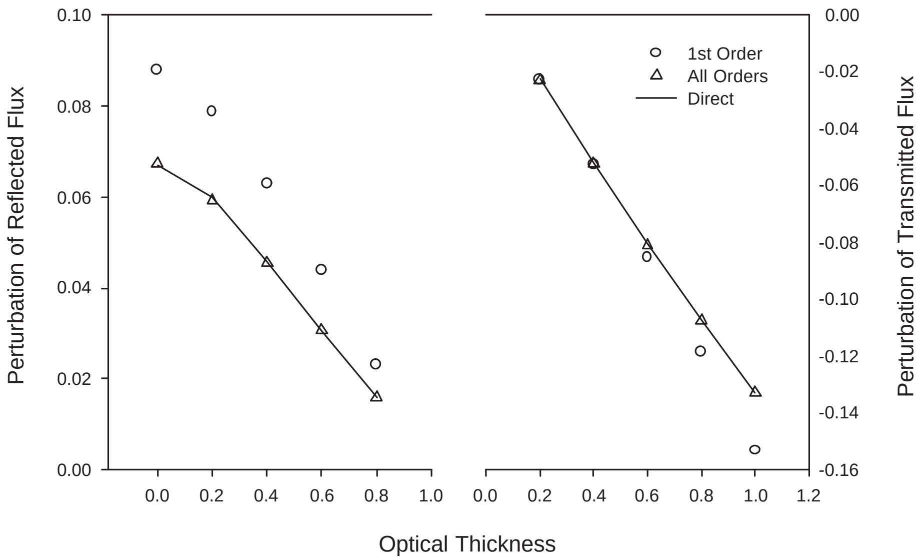

Fig. 1. The perturbation of the reflected and transmitted flux at a number of altitudes measured by the optical thickness of the base model for Case 1.

In Fig. 1, we show a case of a small base model optical thickness (Case 1). In this case, we introduced perturbation into both the optical thickness and the single scattering albedo. In Figs. 2 and 3, we show cases of a large base model optical thickness (Case 2 and Case 3) and perturbations are introduced into the optical thickness and single scattering albedo, respectively.

These figures show the first-order contribution, the full predicted and the directly computed (or “true”) perturbation of the reflected and transmitted fluxes at a number of altitudes measured by the base model optical thickness. In all of these cases, the predictions match the directly computed perturbations very well if the higher-order terms are included. These results show that in these cases the higher-order terms are essential in order to obtain a precise prediction using perturbation theory.

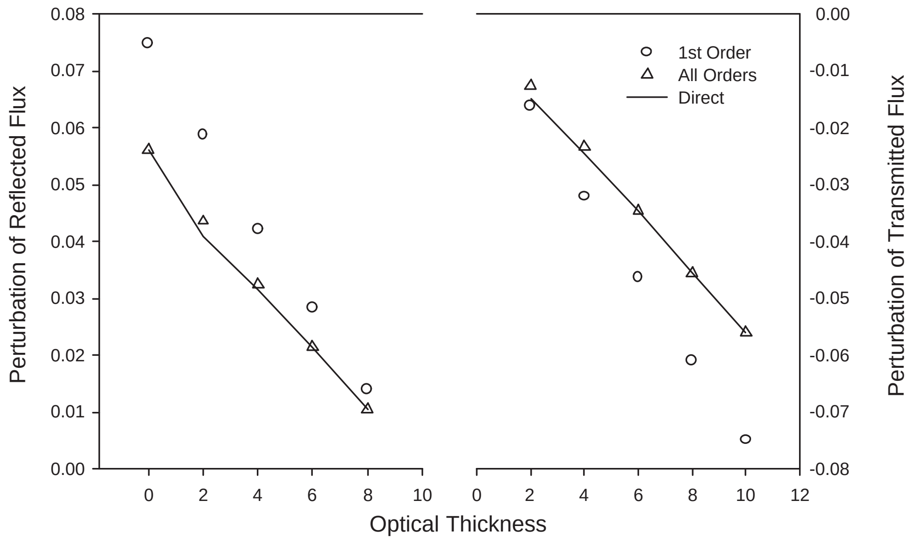

Fig. 2. The perturbation of the reflected and transmitted flux at a number of altitudes measured by the optical thickness of the base model for Case 2.

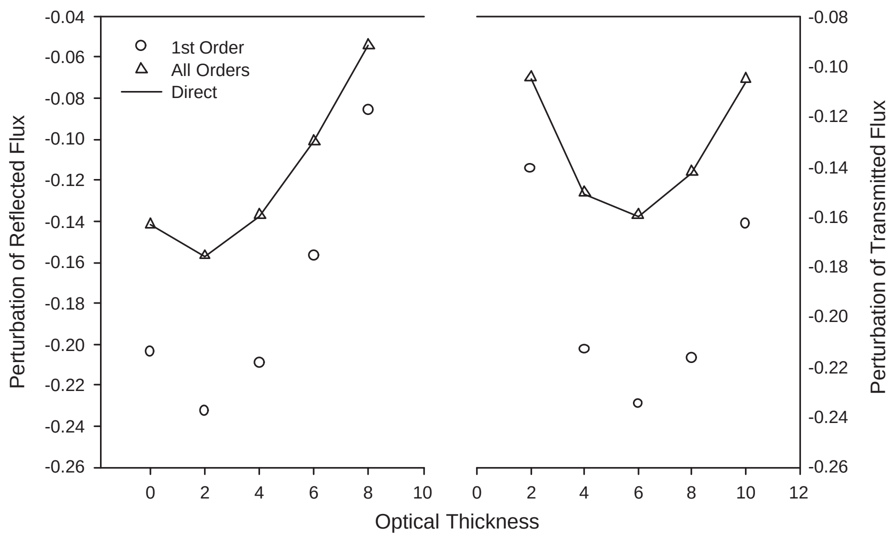

Fig. 3. The perturbation of the reflected and transmitted flux at a number of altitudes measured by the optical thickness of the base model for Case 3.

In Table 2 we list all the orders of the computed perturbation of flux for the three cases considered. (Only the reflected flux at the top of atmosphere and the transmitted flux at the bottom of the atmosphere are listed.) Also shown in this table are the base model fluxes, E_0, the directly computed perturbation, $\Delta E^{(\mathrm{Dir})}$, and the total predicted perturbation, $\Delta E^{(\mathrm{Pred})}$. For Case 1, only four orders are needed for the prediction to be sufficiently accurate. However, the number of orders needed seems

Table 2
The base model fluxes, E_0, the directly computed perturbation, $\Delta E^{(\mathrm{Dir})}$, and the predicted perturbation ($\Delta_1 E$–$\Delta_8 E$ for each order and $\Delta E^{(\mathrm{Pred.})}$ for total) of reflected (at the top of atmosphere) and the transmitted (at the bottom of atmosphere) fluxes for the three considered cases

	Case 1		Case 2		Case 3	
	Reflected	Transmitted	Reflected	Transmitted	Reflected	Transmitted
E_0	1.22E-01	5.85E-01	4.86E-01	2.21E-01	4.86E-01	2.21E-01
$\Delta E^{(\mathrm{Dir})}$	6.71E-02	−1.33E-01	5.60E-02	−5.60E-02	−1.42E-01	−1.06E-01
$\Delta_1 E$	−8.78E-02	1.53E-01	−7.49E-02	7.49E-02	2.04E-01	1.63E-01
$\Delta_2 E$	−2.45E-02	2.11E-02	−2.51E-02	2.51E-02	9.09E-02	8.62E-02
$\Delta_3 E$	−3.43E-03	−9.66E-04	−8.56E-03	8.56E-03	4.27E-02	4.22E-02
$\Delta_4 E$	6.91E-04	−2.46E-03	−2.96E-03	2.96E-03	2.03E-02	2.03E-02
$\Delta_5 E$			−1.06E-03	1.06E-03	9.73E-03	9.72E-03
$\Delta_6 E$			−3.92E-04	3.93E-04	4.66E-03	4.66E-03
$\Delta_7 E$			−1.51E-04	1.51E-04	2.23E-03	2.23E-03
$\Delta_8 E$			−5.94E-05	5.94E-05	1.07E-03	1.07E-03
$\Delta E^{(\mathrm{Pred.})}$	6.74E-02	−1.33E-01	5.61E-02	−5.61E-02	−1.41E-01	−1.05E-01

to increase with the base model optical thickness. In the case that both the base model and the perturbed model are conservative ($\tilde{\omega}_0 = 1.0$) and the surface is black, the net change of the total flux reflected by the atmosphere plus transmitted flux should be zero, which is true for all order terms as shown in Table 2 for Case 2.

4. Summary

In this article, we developed the expressions for the computation of the higher-order radiative perturbation terms for the case of azimuthally independent radiative effects (e.g., fluxes), for plane-parallel atmosphere models. Initial numerical computation has been successfully conducted and results presented.

The initial results show that the computation of the higher-order perturbation is successful. When the base model optical thickness is small, even in the case of very large perturbation, for example 100% optical thickness perturbation in Case 1, the performance of the computation is very good with only a few terms required. When the optical thickness is rather large, for example, $\tau_0 = 10.0$, the performance is also good if the perturbation is not too large. We hope to conduct more computations in the near future to systematically analyse the performance of the computation and to find out the range of its applicability. Results of that study will be reported in a separate article.

References

[1] Box MA. Changes in the surface radiation caused by a scattering layer as calculated using radiative perturbation theory. J Geophys Res 1995;100:11581–4.
[2] Werner HJC. Efficient radiation parameterizations for numerical weather predictions with applications to clouds and aerosols. PhD thesis, School of Physics, University of New South Wales, Australia, 2000.

[3] Box MA, Loughlin PE, Samaras M, Trautmann T. Applications of radiative perturbation theory to changes in absorbing gas. J Geophys Res 1997;102:4333–42.
[4] Loughlin PE, Box MA. Investigating biological response in the UVB as a function of ozone variation using perturbation theory. J Photochem Photobiol B: Biol 1998;43:73–85.
[5] Sendra C, Box MA. Retrieval of the phase function and scattering optical thickness of aerosols: a radiative perturbation theory application. JQSRT 2000;64:499–515.
[6] Qin Y, Box MA, Jupp DLB. Inversion of multi-angle sky radiance measurement for the retrieval of atmospheric optical properties I: algorithm. J Geophys Res 2002;107(D22).
[7] Qin Y, Jupp DLB, Box MA. Inversion of multi-angle sky radiance measurement for the retrieval of atmospheric optical properties II: application. J Geophys Res 2002;107(D22).
[8] Box MA. Radiative perturbation theory: a review. Environ Model Software 2002;17:95–106.
[9] Box MA, Keevers M, McKellar BHJ. On the perturbation series for radiative effects. JQSRT 1988;39:219–23.
[10] Trautmann T, Box MA. Greyn's function computation in radiative transfer theory. In: de Groot RA, Nadrchal J, editors. Physics computing '92. Singapore: World Scientific, 1993. p. 485–7.
[11] Box MA, Trautmann T. An algorithm for higher order perturbation theory in radiative transfer calculations. In: de Groot RA, Nadrchal J, editors. Physics computing '92. Singapore: World Scientific, 1993. p. 280–2.
[12] Box MA, Gerstl SAW, Simmer C. Computation of atmospheric radiative effects via perturbation theory. Beitr Phys Atmos 1989;62:193–9.
[13] Box MA, Polonsky IN, Davis AB. Higher order perturbation theory applied to radiative transfer in non-plane-parallel media. JQSRT 2002;78:105–18.
[14] Qin Y, Box MA, Douriaguine P. Computation of Green's function for radiative transfer. JQSRT 2002; doi:10.1016/S0022-4073(03)00139-0.
[15] Chandrasekhar S. Radiative transfer. New York: Dover, 1965.
[16] Liou KN. A numerical experiment on Chandrasekhar's Discrete-Ordinate method for radiative transfer: application to cloudy and hazy atmospheres. J Atmos Sci 1973;30:1303–26.
[17] Liou KN. Application of the discrete-ordinate method for radiative transfer to inhomogeneous aerosol atmospheres. J Geophys Res 1975;80:3434–40.
[18] Stamnes K, Swanson RA. A new look at the Discrete-Ordinate method for radiative transfer calculations in anisotropically scattering atmospheres. J Atmos Sci 1981;38:387–99.
[19] Stamnes K, Conklin P. A new multi-layer discrete ordinate approach to radiative transfer in vertically inhomogeneous atmospheres. JQSRT 1984;31:273–82.
[20] Qin Y, Jupp DLB, Box MA. Extension of the Discrete-Ordinate algorithm and efficient radiative transfer calculation. JQSRT 2002;74:767–81.
[21] Kattawar GW. A three-parameter analytic phase function for multiple scattering calculations. JQSRT 1975;15:839–49.