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Extended discrete-ordinate method considering full polarization state

Michael A. Box*, Yi Qin[1]

School of Physics, University of New South Wales, Australia

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Abstract

This paper presents an extension to the standard discrete-ordinate method (DOM) to consider generalized sources including: beam sources which can be placed at any (vertical) position and illuminate in any direction, thermal emission from the atmosphere and angularly distributed sources which illuminate from a surface as continuous functions of zenith and azimuth angles. As special cases, the thermal emission from the surface and deep space can be implemented as angularly distributed sources. Analytical-particular solutions for all source types are derived using the infinite medium Green's function. Radiation field zenith angle interpolation using source function integration is developed for all source types. The development considers the full state of polarization, including the sources (as applicable) and the (BRDF) surface, but the development can be reduced easily to scalar problems and is ready to be implemented in a single set of code for both scalar and vector radiative transfer computation.

Keywords: Polarization; Green's function; Discrete ordinate method

1. Introduction

The discrete-ordinate method [1] has been a very successful numerical method for radiative transfer computation since the development of DISORT by Stamnes et al. [2]. The vector DOM

*Corresponding author. Tel.: +61 2 9385 4545; fax: +61 2 9385 6060.
E-mail addresses: m.box@unsw.edu.au (M.A. Box), yi.qin@csiro.au (Y. Qin).
[1]Currently with the Environment Remote Sensing Group, CSIRO Land and Water, GPO Box 1666, Canberra ACT 2601, Australia.

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algorithm or spherical harmonics method, which considers the polarization state has also been developed [3–8] (and works cited there) and benchmarks [5,6] have been published. In particular, Siewert and co-workers successfully developed the (analytical) expansion for the scattering phase matrix which is a key step of the vector DOM. A vector version of DISORT has also been developed [9–11]. All these works assume that the beam source as located at the top of the atmosphere and illuminates downwards, which is sufficient for normal forward radiative transfer computation for the solar source. Of sources, thermal emission from the atmosphere, the surface and the top (deep space) are also considered by DISORT and VDISORT.

In some cases, for example in inversion process [12–16] using adjoint formulation and perturbation theory [17–19], a (adjoint) beam source may be located anywhere in the atmosphere and it may illuminate in any direction. For such cases, an extension to the standard DOM algorithm is required. The purpose of this paper is to extend the vector DOM to consider generalized/extended source types, including generalized beam sources, thermal emission from atmosphere, and an angularly distributed source, which is a generalization of the thermal emission from the surface and deep space.

In a previous paper [20], referred to as paper 1, we presented a Green's function algorithm which shares some steps with the DOM. In this paper, we will refer to paper 1 for information of the shared portion. As in paper 1, although this paper considers the full polarization state, the development can easily be reduced to the case of the scalar problem, which allows us to implement the algorithms in a single set of code for both scalar and vector radiative transfer computations. We firstly define the problem, in particular the source types, in Section 2. The DOM algorithm is presented in Sections 3–5, including extension of the particular solutions. The computation of radiation field at arbitrary zenith angles, which is also source type dependent, will be presented in Section 6. This paper is summarized in Section 7.

2. Radiative transfer problem

In this section, we define the problem to be solved, including the radiative transfer equation, the extended/generalized sources, and the boundary conditions.

2.1. The radiative transfer equation

The radiative transfer problem in a homogeneous layer can be described by the radiative transfer equation [1,21]

$$\mu \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}(\tau,\mu,\phi) = \mathbf{I}(\tau,\mu,\phi) - \frac{\tilde{\omega}_0}{4\pi}\int_{-1}^{1}\mathrm{d}\mu' \int_0^{2\pi}\mathrm{d}\phi' \mathbf{M}(\mu,\phi;\mu',\phi')\mathbf{I}(\tau,\mu',\phi') - \mathbf{Q}(\tau,\mu,\phi), \tag{1}$$

where $\mathbf{I}$ is the 4-vector of the Stokes parameters, I, Q, U and V; (μ,ϕ) are, respectively, the cosine of the zenith angle and the azimuth angle, with $\mu>0$ for upward directions; τ is the optical thickness which is defined to be 0 at the top of the atmosphere and $\tilde{\omega}_0$ is the single scattering albedo. $\mathbf{M}$ is the scattering matrix which, as discussed in detail in Section A3 of paper 1, represents scattering by randomly oriented non-spherical particles with a symmetric plane; $\mathbf{Q}$ is the source function which depends on the type of the source and will be detailed in Section 2.2.

We expand the intensity vector, $\mathbf{I}$, and the source function, $\mathbf{Q}$, into Fourier series in azimuth angle:

$$\mathbf{I}(\tau,\mu,\phi) = \sum_{m=0}^{2N-1}\sum_{\alpha=1}^{2} \mathbf{Z}_\alpha^m(\phi)\mathbf{I}_\alpha^m(\tau,\mu), \tag{2}$$

$$\mathbf{Q}(\tau,\mu,\phi) = \sum_{m=0}^{2N-1}\sum_{\alpha=1}^{2} \mathbf{Z}_\alpha^m(\phi)\mathbf{Q}_\alpha^m(\tau,\mu), \tag{3}$$

where $\mathbf{Z}_\alpha^m$ is diagonal and composed of $\cos(m\phi)$ and $\sin(m\phi)$as defined in Section A3 of paper 1. By expanding also the scattering matrix, $\mathbf{M}$, as shown by Eq. (A48) of paper 1, and following the same process as described in Sections A4, 2.2 and 2.3 of paper 1, Eq. (1) is reduced to a set of independent equations in (τ,μ):

$$\mu\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}_\alpha^m(\mu_i) = \mathbf{I}_\alpha^m(\mu_i) - \sum_{j=\pm1}^{\pm N_s}\frac{\tilde{\omega}_0}{2}w_j\sum_{l=m}^{2N-1}\mathbf{P}_l^m(\mu_i)\mathbf{B}_l^m\mathbf{P}_l^m(\mu_j)\mathbf{I}_\alpha^m(\mu_j) - \mathbf{Q}_\alpha^m(\tau,\mu_i),$$
$$\alpha = 1,2,\, i,j = \pm1,\pm2,\ldots,\pm N_s, \tag{4}$$

where $\mathbf{P}_\alpha^m$ and $\mathbf{B}_\alpha^m$ are defined in Section A3 of paper 1 and we have replaced the integration over μ' with Gaussian quadrature of order $2N_s$ with the quadrature weights and points denoted by w_j and μ_j.

2.2. The source types

The first source type dealt with in this paper is a generalized parallel beam, which extends horizontally indefinitely and may or may not be polarized initially. We define it by

$$\mathbf{Q}(\tau,\mu,\phi) = \delta(\tau-\tau_0)\delta(\mu-\mu_0)\delta(\phi-\phi_0)\mathbf{F}_0, \tag{5}$$

where (τ_0,μ_0,ϕ_0) are the source location and direction (but we may always assume $\phi_0 = 0$); $\mathbf{F}_0$ denotes the source radiation, which may be a 4×4 identity matrix, or a 4-vector of the Stokes parameters. If $\mathbf{F}_0$ is an identity matrix, the source is referred to as generalized beam source (GBS), or otherwise it is simply beam source (BS). By using GBS, the specification of the source can be deferred until the radiative transfer equation is solved, which is more flexible. The GBS also plays an important role in the validation of the VGDOM code as will be discussed in a separate paper (in preparation).

The definition of the (G)BS in Eq. (5), however, is not suitable for numerical computation due to the Dirac delta functions in Eq. (5). Therefore, the diffuse (or local scattering) source function is normally used which, after being expanded into a Fourier series of the azimuth angle, is written as (cf. [5,6])

$$\mathbf{Q}_\alpha^m(\tau,\mu) = \frac{t_{dir}(\tau)}{2\pi}\mathbf{\Omega}(\mu)\,\mathbf{d}^m(\mu,\mu_0)\mathbf{\Omega}(\mu_0)\mathbf{D}_\alpha\mathbf{F}_0$$
$$m = 0,\ldots,2N-1, \alpha = 1,2, \tag{6}$$

where $\mathbf{D}_\alpha$ and $\mathbf{\Omega}(\mu)$ are diagonal as defined in Section A3 and Eq. (44) of paper 1, respectively, the functions t_{dir} and $\mathbf{d}^m$ are defined, respectively, as

$$t_{dir}(\tau) = H\left(\frac{\tau_0 - \tau}{\mu_0}\right)\mathrm{e}^{(\tau-\tau_0)/\mu_0}, \tag{7}$$

$$\mathbf{d}^m(\mu, \mu_0) = \begin{cases} \frac{\tilde{\omega}_0}{2}\sum\limits_{l=m}^{2N-1} \mathbf{P}_l^m(|\mu|)\mathbf{B}_l^m\mathbf{P}_l^m(|\mu_0|) & \mu\mu_0 > 0, \\ \frac{\tilde{\omega}_0}{2}\sum\limits_{l=m}^{2N-1} (-1)^{l-m}\mathbf{P}_l^m(|\mu|)\mathbf{D}\mathbf{B}_l^m\mathbf{P}_l^m(|\mu_0|) & \mu\mu_0 < 0, \end{cases} \tag{8}$$

where $\mathbf{D}$ is defined in Section A3 of paper 1, $|x|$ is the absolute value of x, and H is the Heaviside step function.

The second source type dealt with in this paper is the atmosphere thermal emission source (ATS), which is isotropic and unpolarized. We approximate the emission as a power series in the optical thickness [2]:

$$Q(\tau, \mu, \phi) = (1 - \tilde{\omega}_0)B[T(\tau)] \approx \sum_{k=0}^{N_k} b_k\tau^k, \tag{9}$$

where B is the Planck function and N_k is the order of the power series. The azimuth expansion of the ATS can be written as

$$\mathbf{Q}_\alpha^m(\tau, \mu) = \delta_{0,m}\delta_{1,\alpha}\mathbf{\Omega}(\mu)\mathbf{e}_1\sum_{k=0}^{N_k} b_k^l\tau^k, \tag{10}$$

where δ_{ij} is the Kronecker delta symbol, $\mathbf{e}_1 = (1, 0, 0, 0)^{\mathrm{T}}$ and $\mathbf{\Omega}(\mu)$ is introduced for consistency with the other sources.

Another source type that is dealt with in this paper is called the angularly distributed source (ADS), which is defined as

$$\mathbf{Q}(\tau, \mu, \phi) = \delta(\tau - \tau_0)\sum_{m=0}^{2N-1}\sum_{\alpha=1}^{2}\mathbf{z}_\alpha^m(\phi)\mathbf{\Omega}(\mu)\hat{\mathbf{Q}}_\alpha^m(\mu). \tag{11}$$

This source represents any illumination from a surface, $\tau = \tau_0$, and as a continuous function of μ and ϕ, in either or both of the hemispheres. Cases of ADS include the thermal emission from the surface or illumination/emission from deep space. By writing the definition of ADS as Eq. (11), the azimuth expansion of ADS is

$$\mathbf{Q}_\alpha^m(\tau, \mu) = \delta(\tau - \tau_0)\mathbf{\Omega}(\mu)\hat{\mathbf{Q}}_\alpha^m(\mu). \tag{12}$$

2.3. Boundary and continuity conditions

A set of boundary and continuity conditions should also be satisfied which, when expanded into Fourier series of azimuth angle, can be written as

$$\mathbf{I}_\alpha^{m,1}(0, \boldsymbol{\mu}_-) = \mathbf{0}, \tag{13}$$

$$\mathbf{I}_\alpha^{m,p-1}(\tau^{p-1}, \boldsymbol{\mu}) = \mathbf{I}_\alpha^{m,p}(\tau^{p-1}, \boldsymbol{\mu}) \quad p = 2, 3, \ldots, N_p, \tag{14}$$

$$\mathbf{I}_\alpha^{m,N_p}(\tau_a, \boldsymbol{\mu}_+) = 2\mathbf{R}^m(\boldsymbol{\mu}_+, \boldsymbol{\mu}_-)\boldsymbol{\Omega}(\boldsymbol{\mu}_-)\mathbf{w}_+\boldsymbol{\mu}_+\mathbf{I}_\alpha^{m,N_p}(\tau_a, \boldsymbol{\mu}_-) + \frac{1}{\pi}|\mu_0|t_{dir}(\tau_a)\mathbf{R}^m(\boldsymbol{\mu}_+, \mu_0)\boldsymbol{\Omega}(\mu_0)\mathbf{D}_\alpha\mathbf{F}_0, \tag{15}$$

where $\alpha = 1, 2,\ m = 0, 1, \ldots, 2N-1$ and an extra superscript, p, is used to denote the layer number. The second term on the right-hand side of the last equation exists only for downward beam source. The surface BRDF matrix and its expansion, $\mathbf{R}^m$, is discussed in Section 4 of paper 1. $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_\pm$ as function arguments are interpreted as explained by Eq. (85) of paper 1, $\mathbf{w}$ and $\boldsymbol{\mu}_\pm$ (as non-argument) are defined in Section 2.4 of paper 1. For simplicity, the superscript, m, will be omitted hereafter.

3. The general solution

The solution to Eq. (4) is found by firstly solving its homogeneous version, for which we seek a solution of the following form:

$$\begin{bmatrix} \mathbf{I}(\tau, \boldsymbol{\mu}_+) \\ \mathbf{I}(\tau, \boldsymbol{\mu}_-) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}(\boldsymbol{\mu}_+) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}(\boldsymbol{\mu}_-) \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}(\boldsymbol{\mu}_+) \\ \boldsymbol{\phi}(\boldsymbol{\mu}_-) \end{bmatrix} \mathrm{e}^{-\boldsymbol{\lambda}\tau}, \tag{16}$$

where $\boldsymbol{\phi}(\boldsymbol{\mu}_\pm)$ are $4N_s \times 4$ matrices and $\boldsymbol{\lambda}$ is 4×4 diagonal. Inserting Eq. (16) into Eq. (4) leads to a eigenvalue problem which can be solved as described in Section 2.3 of paper 1. Together with the particular solutions to be obtained in the next section, the complete solution to Eq. (4) can be constructed.

4. The particular solution

The particular solution can be obtained using the infinite medium Green's function [5,6,16,22,23]. In [23], the particular solutions for all the source types have been developed for the scalar radiative transfer problem. Those results can be extended to the vector problem in a parallel way. For simplicity, we only list the final results in this section.

4.1. Particular solution for beam sources

The particular solution for beam sources can be written as

$$\mathbf{I}_{p,\alpha}(\tau,\boldsymbol{\mu}) = \boldsymbol{\Omega}(\boldsymbol{\mu})\hat{\mathbf{I}}_p(\tau,\boldsymbol{\mu})\boldsymbol{\Omega}(\mu_0)\mathbf{D}_\alpha\mathbf{F}_0, \tag{17}$$

where $\hat{\mathbf{I}}_p$ is $8N_s \times 4$ matrix which, when the source does not coincide with any of the Gaussian quadrature points, is defined as

$$\hat{\mathbf{I}}_p(\tau,\boldsymbol{\mu}) = \begin{cases} \boldsymbol{\Phi}\mathrm{e}^{-(\tau_0-\tau)/\mu_0}\mathbf{z} & (\tau_0-\tau)/\mu_0 \geqslant 0, \\ \boldsymbol{\Phi}\mathrm{e}^{\boldsymbol{\lambda}(\tau_0-\tau)}\mathbf{z} & (\tau_0-\tau)/\mu_0 < 0, \end{cases} \tag{18}$$

where $\mathbf{z}$ is $8N_s \times 4$ matrix defined as

$$\mathbf{z} = [(1/\mu_0 + \boldsymbol{\lambda})(-2\pi\mathbf{N})]^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{w}\mathbf{d}(\boldsymbol{\mu},\mu_0), \tag{19}$$

where $\mathbf{N}$, $\boldsymbol{\lambda}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ and $\mathbf{w}$ are defined in Sections 2.3 and 2.4 of paper 1, and $\mathbf{d}$ is defined by Eq. (8).

When a beam source coincides with one of the Gaussian quadrature points, μ_j, $j = \pm 1, \ldots, \pm N_s$, "divide by zero" may occur in Eq. (19) because λ_j may be very close to $-1/\mu_0$ [23]. $\hat{\mathbf{I}}_p$ for this case is found to be

$$\hat{\mathbf{I}}_p = \begin{cases} \boldsymbol{\xi}\mathrm{e}^{-(\tau-\tau_0)/\mu_0} & (\tau_0-\tau)/\mu_0 > 0, \\ \boldsymbol{\Phi}\mathrm{e}^{\boldsymbol{\lambda}(\tau_0-\tau)}\mathbf{z} & (\tau_0-\tau)/\mu_0 < 0, \end{cases} \tag{20}$$

where

$$\boldsymbol{\xi} = (-2\pi\mathbf{w})^{-1}\mathbf{E}_j, \tag{21}$$

$$\mathbf{z} = (-2\pi\mathbf{N})^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\mu}\mathbf{E}_j, \tag{22}$$

where $\mathbf{E}_j$ is a $8N_s \times 4$ matrix whose jth 4×4 sub-matrix is an identity matrix and all other elements are zero. Note that the case for $(\tau_0 - \tau)/\mu_0 < 0$ was not considered in Qin et al. [23], which is completed here.

4.2. Particular solution for atmosphere thermal sources

The particular solution for ATS is found to be:

$$\mathbf{I}_{p,\alpha}(\tau,\boldsymbol{\mu}) = \delta_{0m}\delta_{\alpha,1}\boldsymbol{\Omega}(\boldsymbol{\mu})\begin{bmatrix} \boldsymbol{\Phi}_+ & \boldsymbol{\Phi}_- \end{bmatrix} \begin{bmatrix} \mathbf{h}_+(\tau) & \\ & \mathbf{h}_-(\tau) \end{bmatrix} \begin{bmatrix} -\mathbf{N}_+^{-1}{}_+\boldsymbol{\Psi}^{\mathrm{T}} \\ -\mathbf{N}_+^{-1}{}_-\boldsymbol{\Psi}^{\mathrm{T}} \end{bmatrix} \mathbf{w}\mathbf{E}_1, \tag{23}$$

where $\mathbf{E}_1$ is defined following Eq. (22), and $\mathbf{h}_\pm$ are defined as

$$\mathbf{h}_\pm(\tau) = \sum_{n=0}^{N_k} \mathbf{a}_{\pm n}\tau^n, \tag{24}$$

where

$$\mathbf{a}_{\pm n} = \pm \sum_{k=n}^{N_k} (-1)^{k-n} \frac{k!}{n!} b_k \boldsymbol{\lambda}_{\pm}^{-(k-n+1)}. \tag{25}$$

4.3. Particular solution for angularly distributed sources

The particular solution for ADS is found to be:

$$\mathbf{I}_{p,\alpha}(\tau, \boldsymbol{\mu}) = \boldsymbol{\Omega}(\boldsymbol{\mu})\boldsymbol{\Phi}_{\pm}\boldsymbol{\Lambda}_{\pm}(\tau, \tau_0)\mathbf{z}_{\alpha}, \tag{26}$$

$$\mathbf{z}_{\alpha} = -\mathbf{N}_{+}^{-1}{}_{\pm}\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{w}|\boldsymbol{\mu}|\hat{\mathbf{Q}}_{\alpha}^{m}(\boldsymbol{\mu}), \tag{27}$$

where $\boldsymbol{\Lambda}_{\pm}(\tau, \tau_0)$ is defined by Eq. (65) of paper 1, $\hat{\mathbf{Q}}_{\alpha}^{m}(\boldsymbol{\mu}_{\pm})$ are defined by Eq. (12), and the subscript, "$\pm$", correspond to $\tau > \tau_0$ and $\tau < \tau_0$, respectively (see Section 2.4 of paper 1). For the case of thermal emission from the surface, $\hat{\mathbf{Q}}_{\alpha}^{m}(\boldsymbol{\mu}_{\pm})$ is

$$\hat{\mathbf{Q}}_{\alpha}^{m}(\mu) = \delta_{0m}\delta_{\alpha,1}B(T_s)A_{\mathrm{e}}(\mu)\mathbf{e}_1 \quad \mu > 0, \tag{28}$$

where $A_{\mathrm{e}}(\mu)$ is the angular emissivity of the surface. For thermal emission from deep space,

$$\hat{\mathbf{Q}}_{\alpha}^{m}(\mu) = \delta_{0m}\delta_{\alpha,1}B(T_t)\varepsilon_t\mathbf{e}_1 \quad \mu < 0, \tag{29}$$

where ε_t is the (isotropic) emissivity.

5. The complete solution

We can generally write the complete solution to Eq. (4) for layer p, disregarding the source type as

$$\begin{aligned} \mathbf{I}_{\alpha}^{p}(\tau, \mu) &= \boldsymbol{\Omega}(\mu)\hat{\mathbf{I}}_{\alpha}^{p}(\tau, \mu), \\ \hat{\mathbf{I}}_{\alpha}^{p}(\tau, \mu) &= \boldsymbol{\Gamma}^{p}(\tau, \mu)\mathbf{Y}_{\alpha}^{p} + \hat{\mathbf{I}}_{p,\alpha}^{p}(\tau, \mu), \end{aligned} \tag{30}$$

where $\boldsymbol{\Gamma}^{p}(\tau, \mu)$ is defined in Section 3.2 of paper 1. $\mathbf{Y}_{\alpha}^{p}$ are integral constants which are yet to be determined from the boundary and continuity conditions, Eqs. (13)–(15). The superscript, p, is attached to all quantities that depend on the layer.

Applying Eq. (30) to the boundary conditions, Eqs. (13)–(15), we obtain:

$$-\boldsymbol{\Gamma}^{1}(0, \boldsymbol{\mu}_{-})\mathbf{Y}_{\alpha}^{1} = \hat{\mathbf{I}}_{p,\alpha}^{1}(0, \boldsymbol{\mu}_{-}), \tag{31}$$

$$\begin{aligned} &\boldsymbol{\Gamma}^{p-1}(\tau^{p-1}, \boldsymbol{\mu})\mathbf{Y}_{\alpha}^{p-1} - \boldsymbol{\Gamma}^{p}(\tau^{p-1}, \boldsymbol{\mu})\mathbf{Y}_{\alpha}^{p} = \hat{\mathbf{I}}_{p,\alpha}^{p}(\tau^{p-1}, \boldsymbol{\mu}) - \hat{\mathbf{I}}_{p,\alpha}^{p-1}(\tau^{p-1}, \boldsymbol{\mu}), \\ &\quad p = 2, 3, \ldots, N_p, \end{aligned} \tag{32}$$

$$(\mathbf{\Gamma}^{N_p}(\tau_a, \boldsymbol{\mu}_+) - 2\mathbf{R}(\boldsymbol{\mu}_+, \boldsymbol{\mu}_-)\mathbf{w}_+\boldsymbol{\mu}_+\mathbf{\Gamma}^{N_p}(\tau_a, \boldsymbol{\mu}_-))\mathbf{Y}^{N_p} = -\hat{\mathbf{I}}_p^{N_p}(\tau_a, \boldsymbol{\mu}_+) + 2\mathbf{R}(\boldsymbol{\mu}_+, \boldsymbol{\mu}_-)\mathbf{w}_+\boldsymbol{\mu}_+\hat{\mathbf{I}}_p^{N_p}(\tau_a, \boldsymbol{\mu}_-) + \frac{1}{\pi}|\mu_0|t_{\mathrm{dir}}(\tau_a)\mathbf{R}(\boldsymbol{\mu}_+, \mu_0)\mathbf{\Omega}(\mu_0)\mathbf{D}_\alpha\mathbf{F}_0, \tag{33}$$

where the second term on the right-hand side of Eq. (33) exists only for downward beam sources. Eqs. (31)–(33) form linear equation system with a band diagonal coefficient matrix.

6. Intensity at arbitrary zenith angles

The previous sections presented the computation of the radiation field at the zenith angles specified by the Gaussian quadrature points, μ_i, $i = \pm 1, \ldots, \pm N_s$. This section presents the method used to interpolate the intensity at arbitrary zenith angles.

Two schemes, namely dummy quadrature node (DQN) and source function integration (SFI), are commonly used to perform angular interpolation [24]. Although they have found that the DQN scheme might be faster than the SFI scheme, the SFI scheme is more flexible in that it is independent of the original algorithm. In this section, the SFI method is extended to all the source types considered in this paper.

The SFI method essentially uses the integral form of the radiative transfer equation, which can be derived from Eq. (4). Firstly we define

$$\mathbf{J}(\tau, \mu) = \sum_{j=\pm 1}^{\pm N_s} \mathbf{K}(\mu, \mu_j)\mathbf{\Omega}(\mu_j)\mathbf{I}(\tau, \mu_j) + \mathbf{Q}(\tau, \mu), \tag{34}$$

where **I**, **Q** and **J** all depend on m and α, which are omitted for simplicity, and

$$\mathbf{K}(\mu, \mu_j) = \begin{cases} \frac{\tilde{\omega}_0}{2} w_j \sum\limits_{l=m}^{2N-1} \mathbf{P}_l^m(\mu)\mathbf{B}_l^m\mathbf{P}_l^m(|\mu_j|) & \text{for} \quad j>0, \\ \frac{\tilde{\omega}_0}{2} w_j \sum\limits_{l=m}^{2N-1} (-1)^{l-m}\mathbf{P}_l^m(\mu)\mathbf{B}_l^m\mathbf{D}\mathbf{P}_l^m(|\mu_j|) & \text{for} \quad j<0. \end{cases} \tag{35}$$

Using the definition of **J** and multiplying both sides of Eq. (4) with $e^{-\tau/\mu}$, we obtain:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathbf{I}(\mu_i)e^{-\tau/\mu}) = -\mathbf{J}(\tau, \mu_i)e^{-\tau/\mu}. \tag{36}$$

Integrate both sides from 0 to τ, and recall the upper boundary condition, $\mathbf{I}(0, \mu) = \mathbf{0}$, we obtain for $\mu<0$:

$$\mathbf{I}(\tau, \mu) = \int_0^\tau |\mu^{-1}|\mathbf{J}(\tau', \mu)e^{-(\tau'-\tau)/\mu}\,\mathrm{d}\tau' \quad \text{for} \;\; \mu<0. \tag{37}$$

Similarly for $\mu>0$ we have:

$$\mathbf{I}(\tau, \mu) = \int_\tau^{\tau_a} |\mu^{-1}|\mathbf{J}(\tau', \mu)e^{-(\tau'-\tau)/\mu}\,\mathrm{d}\tau' + \mathbf{I}_\alpha^m(\tau_a, \mu)e^{-(\tau_a-\tau)/\mu} \quad \text{for} \;\; \mu>0. \tag{38}$$

Because $\mathbf{I}(\tau, \mu_j)$, where μ_j are the Gaussian points, have been obtained in the previous sections, and we assume that $\mathbf{Q}(\tau, \mu)$ is known for any value of μ, Eqs. (37) and (38) provide the formulation to compute $\mathbf{I}(\tau, \mu)$ at arbitrary zenith angles. Using Eq. (30) we can write $\mathbf{J}(\tau, \mu)$ as

$$\mathbf{J}(\tau, \mu) = {}_{\mu}\mathbf{K}\boldsymbol{\Gamma}(\tau, \boldsymbol{\mu})\mathbf{Y} + {}_{\mu}\mathbf{D}\hat{\mathbf{I}}_p(\tau, \boldsymbol{\mu}) + \mathbf{Q}(\tau, \mu), \tag{39}$$

where ${}_{\mu}\mathbf{K} = \{\mathbf{K}(\mu, \mu_j), j = \pm 1, \ldots, \pm N_s\}$ is a $4 \times 8N_s$ matrix. We note that a subscript, μ, is pre-attached to $\mathbf{K}$ and $\mathbf{D}$ to show the output zenith angle.

Inserting Eq. (39) into Eqs. (37) and (38) we will have three terms for the case of $\mu < 0$. We refer to these terms, respectively, as the general solution term, the particular solution term and the source term. For the case of $\mu > 0$ we have, in addition to these three terms, the last term in Eq. (38), which is referred to as the boundary term. The general solution term is the same for all source types, and will be presented in Section 6.1. The other three terms depend on the source type, and will be presented in Sections 6.2–6.4.

6.1. The general solution term

Insert the first term of Eq. (39) into Eqs. (37) and (38), and carry out the integration, we obtain the general solution term for all source types:

$$\mathbf{I}_g(\tau, \mu) = \sum_{p=p_1}^{p_2} ({}_{\mu}\mathbf{K}^p \boldsymbol{\Phi}^p \mathbf{h}^p(\tau) \mathbf{Y}^p), \tag{40}$$

where again the superscript, p, denotes layer number and

$$\mathbf{h}^p(\tau) = \int_{\tau_1^p}^{\tau_2^p} |\mu^{-1}| \boldsymbol{\Lambda}^p(\tau') \mathrm{e}^{-(\tau' - \tau)/\mu} \, \mathrm{d}\tau' = diag(\mathbf{h}_+^p, \mathbf{h}_-^p) \tag{41}$$

and p_1, p_2 and other parameters are summarized in Table 1, and

$$\mathbf{h}_\pm^p(\tau) = (\mathrm{sgn}(\mu) + |\mu| \boldsymbol{\lambda}_\pm^p)^{-1} \left(\mathrm{e}^{\boldsymbol{\lambda}_\pm^p(\tau_{\pm b} - \tau_1^p) + (\tau - \tau_1^p)/\mu} - \mathrm{e}^{\boldsymbol{\lambda}_\pm^p(\tau_{\pm b} - \tau_2^p) + (\tau - \tau_2^p)/\mu} \right) \tag{42}$$

are also diagonal matrices, where the second factor of $\mathbf{h}_\pm^p$ is further translated into Table 2, for computational convenience, after the variable parameters being substituted by the values defined in Table 1. When $\lambda_{\pm j}^p = -1/\mu$, the $\pm j$th diagonal element of $\mathbf{h}_\pm^p$ can be computed using

$$h_{\pm j}^p = |\mu^{-1}| (\tau_2^p - \tau_1^p) \mathrm{e}^{\lambda_{\pm j}^p \tau_{\pm b} + \tau/\mu}. \tag{43}$$

6.2. Specific terms for beam sources

In this subsection we present the expressions for the particular solution term, the source term and the boundary term for beam sources.

6.2.1. Particular solution term

The particular solution term for beam sources can be found by using Eqs. (18) or (20) and carrying out the integrations in Eqs. (37) and (38) for the particular solution part of $\mathbf{J}$. However,

Table 1
Parameters related to the integrations for the general solution term

	p_1	p_2	τ_1^p	τ_2^p	τ_{+b}	τ_{-b}
$\mu<0$	1	p_2 satisfies $\tau^{p_2-1}<\tau\leqslant\tau^{p_2}$	τ^{p-1}	$\begin{cases}\tau^p & p<p_2\\ \tau & p=p_2\end{cases}$	τ^{p-1}	τ^p
$\mu>0$	p_1 satisfies $\tau^{p_1-1}\leqslant\tau<\tau^{p_1}$	N_p	$\begin{cases}\tau & p=p_1\\ \tau^{p-1} & p>p_1\end{cases}$	τ^p		

Table 2
The computational equations for the second factor of $\mathbf{h}_\pm$, Eq (42)

	2nd factor, $\mathbf{h}_+^p(\tau)$	2nd factor, $\mathbf{h}_-^p(\tau)$
$\tau\leqslant\tau^{p-1}(\mu>0)$ $\tau\geqslant\tau^p(\mu<0)$	$\mathrm{e}^{(\tau-\tau^{p-1})/\mu}-\mathrm{e}^{-\lambda_+^p\Delta\tau^p+(\tau-\tau^p)/\mu}$	$\mathrm{e}^{-\lambda_+^p\Delta\tau^p+(\tau-\tau^{p-1})/\mu}-\mathrm{e}^{(\tau-\tau^p)/\mu}$
$\tau^{p-1}<\tau<\tau^p$ and $\mu>0$	$\mathrm{e}^{\lambda_+^p(\tau^{p-1}-\tau)}-\mathrm{e}^{-\lambda_+^p\Delta\tau^p+(\tau-\tau^p)/\mu}$	$\mathrm{e}^{-\lambda_+^p(\tau^p-\tau)}-\mathrm{e}^{(\tau-\tau^p)/\mu}$
$\tau^{p-1}<\tau<\tau^p$ and $\mu<0$	$\mathrm{e}^{(\tau-\tau^{p-1})/\mu}-\mathrm{e}^{\lambda_+^p(\tau^{p-1}-\tau)}$	$\mathrm{e}^{-\lambda_+^p\Delta\tau^p+(\tau-\tau^{p-1})/\mu}-\mathrm{e}^{-\lambda_+^p(\tau^p-\tau)}$

In the table $\Delta\tau^p=\tau^p-\tau^{p-1}$.

the particular solution changes if the source is not located right on a layer boundary. In that case, the integration needs be carried out in two sections, referred to, respectively, as dark shadow section and light shadow section as illustrated in Fig. 1. The intervals of the two sections are determined by μ_0, μ, τ_0 and τ, and are summarized in Table 3. We note, however, for each combination of μ_0 and μ, that Table 3 shows both sections but one (or both) of them may actually not apply if the lower limit is larger than the upper limit.

For the light shadow section, the result of the integration is

$$\mathbf{I}_{pf}(\tau,\mu)=\begin{cases}\displaystyle\sum_{p=p_1}^{p_2} h_{pbf}^p({}_\mu\mathbf{K}^p\mathbf{\Phi}^p\mathbf{z}^p) & \mu_0\neq\mu_j,\\ \displaystyle\sum_{p=p_1}^{p_2} h_{pbf}^p({}_\mu\mathbf{K}^p\boldsymbol{\xi}^p) & \mu_0=\mu_j,\end{cases} \tag{44}$$

where p_1,p_2 can be decided from Fig. 1, $\mathbf{z}^p$ and $\boldsymbol{\xi}^p$ are defined by Eqs. (19) and (21) respectively, μ_j are the Gaussian points and

$$h_{pbf}^p=|\mu^{-1}|\mathrm{e}^{\tau/\mu-\tau_0/\mu_0}\int_{\tau_1^p}^{\tau_2^p}\mathrm{e}^{\tau'/\mu_0-\tau'/\mu}\,\mathrm{d}\tau'=(|\mu|/\mu_0-\mathrm{sgn}(\mu))^{-1}$$
$$\times(\mathrm{e}^{(\tau_2^p-\tau_0)/\mu_0+(\tau-\tau_2^p)/\mu}-\mathrm{e}^{(\tau_1^p-\tau_0)/\mu_0+(\tau-\tau_1^p)/\mu}), \tag{45}$$

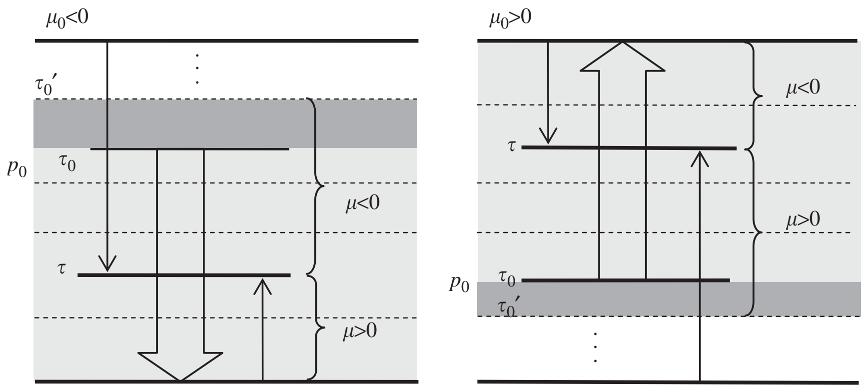

Fig. 1. Integral intervals for the particular solution term of beam sources. Dashed lines are layer-boundaries, τ_0 is the source position, the block arrows indicate the source direction, single arrows denote the radiation propagation direction.

Table 3
Integration intervals for particular solution term of the beam sources

		$\mu_0<0$		$\mu_0>0$	
		τ_{low}	τ_{up}	τ_{low}	τ_{up}
$\mu<0$	DSS	τ_0'	$\min(\tau_0,\tau)$	τ_0	$\min(\tau_0',\tau)$
	LSS	τ_0	τ	0	$\min(\tau_0,\tau)$
$\mu>0$	DSS	$\max(\tau_0',\tau)$	τ_0	$\max(\tau_0,\tau)$	τ_0'
	LSS	$\max(\tau_0,\tau)$	τ_a	τ	τ_0

DSS—dark shadow section, LSS—light shadow section, τ_{low}—lower limit, τ_{up}—upper limit.

where (τ_1^p,τ_2^p) is the common part of (τ^{p-1},τ^p) and the (whole) light shadow section. When $\mu_0=\mu$, Eq. (45) should be replaced by

$$h_{pbf}^p = |\mu^{-1}|(\tau_2^p - \tau_1^p)\,\mathrm{e}^{\tau/\mu+\tau_0/\mu_0}. \tag{46}$$

For the dark shadow section, the result of the integration is

$$\mathbf{I}_{pb}(\tau,\mu) = {}_{\mu}\mathbf{K}^{p_0}\mathbf{w}\mathbf{\Phi}^{p_0}\mathbf{h}_{pbb}^{p_0}\mathbf{z}^{p_0}, \tag{47}$$

where p_0 is the source layer as shown in Fig. 1, $\mathbf{z}^{p_0}$ is defined by Eq. (19) or Eq. (22), and

$$\mathbf{h}_{pbb}^p = (\mathrm{sgn}(\mu) + |\mu|/\boldsymbol{\lambda}^p)^{-1}(\mathrm{e}^{\boldsymbol{\lambda}^p(\tau_0-\tau_1)+(\tau-\tau_1)/\mu} - \mathrm{e}^{\boldsymbol{\lambda}^p(\tau_0-\tau_2)+(\tau-\tau_2)/\mu}), \tag{48}$$

where (τ_1, τ_2) is the dark shadow section defined in Table 3. When $\lambda_j^p = -1/\mu$, Eq. (48) should be replaced by

$$h_{pbb,j}^p = |\mu^{-1}|(\tau_2 - \tau_1)\,e^{\lambda_j^p \tau_0 + \tau/\mu}. \tag{49}$$

We finally obtain the particular solution term for beam sources as the sum of the integrations in the two shadowed sections:

$$\mathbf{I}_p(\tau,\mu) = \mathbf{I}_{pf}(\tau,\mu) + \mathbf{I}_{pb}(\tau,\mu). \tag{50}$$

6.2.2. The source and boundary terms

The source function of the beam sources is defined by Eq. (6). Using this function and carrying out the integration of Eqs. (37) and (38) for the source function part of $\mathbf{J}$, we obtain:

$$\mathbf{I}_q(\tau,\mu) = \sum_{p=p_1}^{p_2} \mathbf{d}^p(\mu) h_{pbf}^p, \tag{51}$$

where (p_1, p_2) covers the range marked by the block arrows in Fig. 1, $\mathbf{d}^p$is defined by Eq. (8), and h_{pbf}^p is the same as defined in Section 6.2.1.

When $\mu > 0$, the reflection from the surface contributes part of the total upward radiation as shown in Eq. (38), which can be computed as

$$\mathbf{I}_{ref}(\tau,\mu) = \mathbf{I}(\tau_a,\mu) e^{(\tau-\tau_a)/\mu}, \tag{52}$$

where $\mathbf{I}(\tau_a,\mu)$ can be computed similarly to Eq. (15) by replacing the argument $\boldsymbol{\mu}_+$ with μ.

6.3. Specific terms for atmosphere thermal source

Using Eq. (23) and carrying out the integration of Eqs. (37) and (38) for the particular solution part of $\mathbf{J}$, we obtain the particular solution term for ATS as

$$\mathbf{I}_p(\tau,\mu) = \sum_{p=p_1}^{p_2} \mu \mathbf{K}^p \boldsymbol{\Phi}^p \sum_{n=0}^{N_k} h_n^p(\tau) \mathbf{z}_n^p, \tag{53}$$

where $h_n^p(\tau)$ is computed recursively as

$$h_n^p(\tau) = \operatorname{sgn}(\mu)((\tau_1^p)^n e^{(\tau-\tau_1^p)/\mu} - (\tau_2^p)^n e^{(\tau-\tau_2^p)/\mu}) + n\mu h_{n-1}^p(\tau), \tag{54}$$

where p_1, p_2, τ_1^p and τ_2^p are summarized in Table 4 and $h_{-1}^p = 0$.

The source term for ATS can be obtained by using Eq. (10) and carrying out Eqs. (37) and (38) for the source function part of $\mathbf{J}$. The result is

$$\mathbf{I}_q(\tau,\mu) = \sum_{p=p_1}^{p_2} \sum_{n=0}^{N_k} h_n^p(\tau) b_n^p \mathbf{e}_1, \tag{55}$$

where p_1, p_2 and $h_n^p(\tau)$ are the same as for Eq. (53).

The boundary term (when $\mu > 0$) for ATS can be computed similarly to beam sources, i.e., Eq. (52), but note that the second term in Eq. (15) does not exist for ATS.

Table 4
Integral interval for atmosphere thermal sources (ATS)

	p_1	p_2	τ_1^p	τ_2^p
$\mu<0$	1	$\tau^{p_2-1}<\tau\leqslant\tau^{p_2}$	τ^{p-1}	$\min(\tau^p,\tau)$
$\mu>0$	$\tau^{p_1-1}\leqslant\tau<\tau^{p_1}$	N_p	$\min(\tau^{p-1},\tau)$	τ^p

Table 5
Integral interval for angular distributed sources (ADS)

	τ_1	τ_{10}	τ_2	τ_{20}
$\mu<0$	τ^{p-1}	$\min(\tau_0,\tau)$	τ_0	$\min(\tau^p,\tau)$
$\mu>0$	$\min(\tau^{p-1},\tau)$	τ_0	$\max(\tau_0,\tau)$	τ^p

6.4. *Specific terms for angularly distributed source*

Using Eq. (26) and carrying out the integration of Eqs. (37) and (38) for the particular solution part of $\mathbf{J}$, we obtain the particular solution term for ADS as

$$\mathbf{I}_p(\tau,\mu) = \begin{pmatrix} {}_\mu\mathbf{K}^p\mathbf{\Phi}_+^p & {}_\mu\mathbf{K}^p\mathbf{\Phi}_-^p \end{pmatrix} \begin{bmatrix} \mathbf{h}_+^p(\tau_{20},\tau_2) & \\ & \mathbf{h}_-^p(\tau_{20},\tau_2) \end{bmatrix} \mathbf{z}^p, \tag{56}$$

where τ_1, τ_{10}, τ_2 and τ_{20} are summarized in Table 5 and

$$\mathbf{h}_\pm^p(t_1,t_2) = (\mathrm{sgn}(\mu) + \boldsymbol{\lambda}_\pm^p|\mu|)^{-1}(\mathrm{e}^{\boldsymbol{\lambda}_\pm^p(\tau_0-t_1)+(\tau-t_1)/\mu} - \mathrm{e}^{\boldsymbol{\lambda}_\pm^p(\tau_0-t_2)+(\tau-t_2)/\mu}). \tag{57}$$

When $\lambda_j^p = -\mu^{-1}$, the jth diagonal element can be computed using

$$h_j^p(t_1,t_2) = |\mu^{-1}|(t_2-t_1)\mathrm{e}^{\lambda_j^p\tau_0+\tau/\mu}. \tag{58}$$

The source term for ADS is obtained similarly as

$$\mathbf{I}_q(\tau,\mu) = \begin{cases} \mathbf{Q}(\mu)\mathrm{e}^{(\tau-\tau_0)/\mu} & \tau_1\leqslant\tau_0\leqslant\tau_2, \\ \mathbf{0} & \text{other}, \end{cases} \tag{59}$$

where τ_1 and τ_2 are shown in Table 5, and the boundary term for ADS is the same as for ATS.

7. Summary

This paper presents an extension to the standard DOM algorithm for radiative transfer in plane-parallel atmospheres. The extended algorithm supports generalized source types, including beam sources (BS) with arbitrary vertical location and illumination directions, atmosphere thermal emission source (ATS), and angularly distributed sources that generalize the thermal emission from the surface or deep space.

The extended algorithm considers the full polarization state (including the polarization of BS and ADS). However, the development can be easily reduced to solve scalar radiative transfer problems, ready to be implemented in a single set of code for both scalar and vector radiative transfer computation. BRDF surface and radiation field at arbitrary zenith angle are supported. The implementation of the algorithm and its validation will be discussed in a separate paper.

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