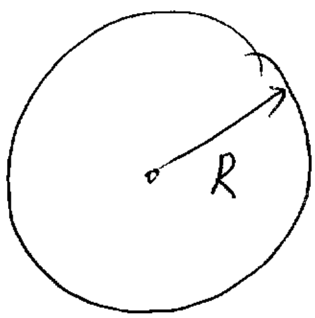
PART IV

Shell model

1) Saxon-Woods and 3D oscillator potentials
2) Quantum states of a 3D oscillator
3) Shells, magic nuclei
4) Spin-orbit interaction
5) Pairing of nucleons

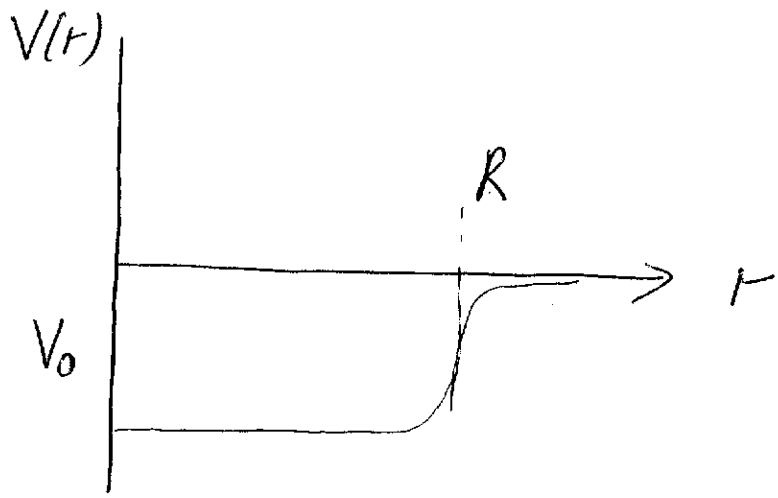
Shell model

Spherical nucleus (drop of nuclear matter)



$$R = r_0 A^{1/3} \quad , \quad r_0 \approx 1.1 \, fm$$

Nuclear selfconsistent potential: the effective potential for nucleons inside the nucleus.



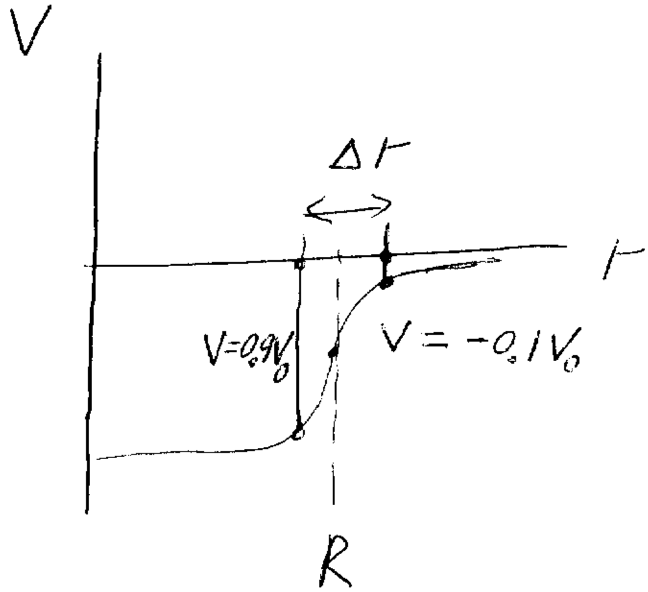
Saxon-Woods potential is an empirical fit of the nuclear potential

$$V(r) = -\frac{V_0}{1 + e^{\frac{r-R}{a}}}$$

$R \approx 1.25\, A^{1/3}$ fm

$a \approx 0.65$ fm

$V_0 \approx 50$ fm



$\Delta r = 4a \ln 3 \approx 2$ fm - "skin thickness"

To find proton and neutron energy levels one has to solve numerically Schrodinger equation

$$\left[\frac{p^2}{2m} + V(r)\right]\psi = E\,\psi$$

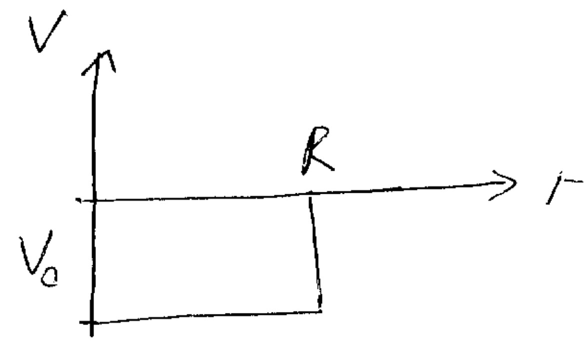
Separation of variables

$$\psi(r,\theta,\varphi) = \frac{1}{r}\chi(r)Y_{\ell m}(\theta,\varphi)$$

$$\left[-\frac{1}{2m}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2mr^2} + V(r)\right]\chi(r) = E\,\chi(r)$$

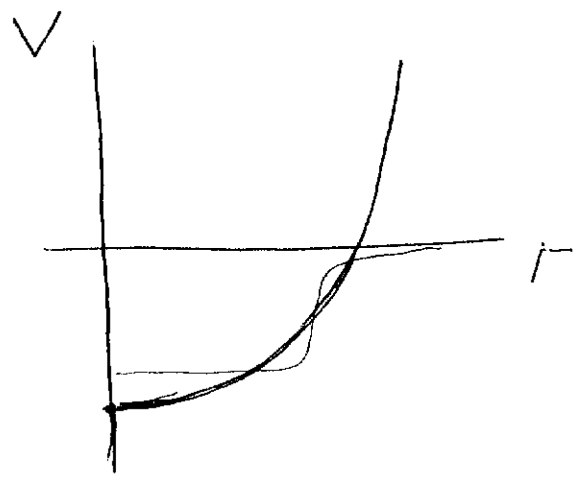
Simplified versions of the effective potential which allow analytical solution

1) 3D square well



2) 3D oscillator

$$V(r) = \frac{m\omega^2 r^2}{2} + \text{const}$$



To find ω:

$$\frac{m\omega^2R^2}{2} \simeq V_0 \Rightarrow \omega = \sqrt{\frac{2V_0}{mR^2}} = \sqrt{\frac{2V_0}{m}}\frac{1}{1.25\,fm\,A^{1/3}}$$

$$= \sqrt{\frac{2\cdot 50}{940}}\frac{197}{1.25}\,Mev\,\frac{1}{A^{1/3}} = \frac{51\,MeV}{A^{1/3}}$$

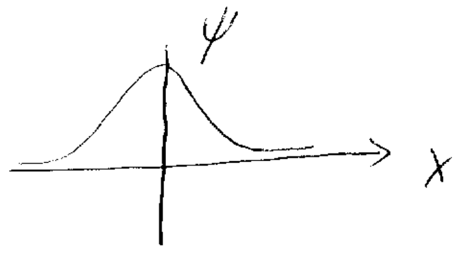
The best fit of energy levels is achieved at

$$\boxed{\omega = \frac{40\,MeV}{A^{1/3}}}$$

<u>1D oscillator</u>, $H = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2}$

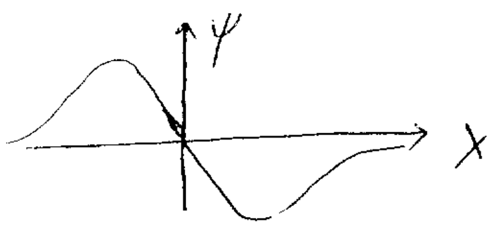
$E_n = \hbar\omega(n+\frac{1}{2})$

$n=0$



$\psi_0 \sim e^{-\alpha x^2}$, $\alpha = \frac{m\omega}{2\hbar}$

$n=1$



$\psi_1 \sim x e^{-\alpha x^2}$

$$\text{Parity} = (-1)^n \Longleftrightarrow \psi_n(-x) = (-1)^n\psi_n(x)$$

3D oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 r^2}{2} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{m\omega^2 y^2}{2} + \frac{m\omega^2 z^2}{2}$$

$$= H_x + H_y + H_z$$

Hence

$$\begin{cases} \psi(x,y,y) = \psi_{n_x}(x)\,\psi_{n_y}(y)\,\psi_{n_z}(z) \\ E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + \frac{1}{2} + n_y + \frac{1}{2} + n_z + \frac{1}{2}\right) = \hbar\omega\left(n + \frac{3}{2}\right) \\ \text{where} \quad n = n_x + n_y + n_z \end{cases}$$

Ground state: $n_x = n_y = n_z = n = 0$

$E_0 = \frac{3}{2}\omega$

$$\psi_{000}(x,y,z) = A e^{-\alpha x^2} e^{-\alpha y^2} e^{-\alpha z^2} = A e^{-\alpha r^2}$$

S-wave state ($\ell = 0$), no dependence on angles.

First excitation, $n=1$, 3 degenerate states:

n_x	n_y	n_z
1	0	0
0	1	0
0	0	1

$$E_1 = \frac{5}{2}\omega$$

$$\begin{cases} \psi_{100} = A x e^{-\alpha x^2} e^{-\alpha y^2} e^{-\alpha z^2} = A x e^{-\alpha r^2} \\ \psi_{010} = A y e^{-\alpha r^2} \\ \psi_{001} = A z e^{-\alpha r^2} \end{cases}$$

$$\text{Parity} = (-1)^{n_x}(-1)^{n_y}(-1)^{n_z} = (-1)^n = -1$$

Spherical coordinates

$$\begin{cases} z = r\cos\theta \\ x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{cases}$$

Basis with fixed angular momentum

$|l, l_z\rangle$, $l = 1$, $l_z = -1, 0, +1$

$$|l=1, l_z=1\rangle = \frac{1}{\sqrt{2}}(-\psi_{100} - i\psi_{010}) \sim (-x - iy)e^{-\alpha r^2} \rightarrow \underbrace{\left(-\sqrt{\frac{3}{8\pi}}\, e^{i\varphi}\sin\theta\right)}_{Y_{11}} A' r e^{-\alpha r^2}$$

$$|l=1, l_z=0\rangle = \psi_{001} \sim \underbrace{\left(\sqrt{\frac{3}{8\pi}}\cos\theta\right)}_{Y_{10}} A' r e^{-\alpha r^2}$$

$$|l=1, l_z=-1\rangle = \frac{1}{\sqrt{2}}(\psi_{100} - i\psi_{010}) \sim \underbrace{\left(\sqrt{\frac{3}{8\pi}}\, e^{-i\varphi}\sin\theta\right)}_{Y_{1-1}} A' r e^{-\alpha r^2}$$

$$\text{Parity} = (-1)^l = -1$$

$n=2$: 6 states with even parity

Cartezian coordinates

n_x	n_y	n_z
2	0	0
0	2	0
0	0	2
1	1	0
1	0	1
0	1	1

$$E = \frac{7}{2}\omega$$

$$P = (-1)^{n_x}(-1)^{n_y}(-1)^{n_z} = (-1)^n = +1$$

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spherical coordinates

$\ell = 2, \quad \ell_z = -2, -1, 0, +1, +2 \rightarrow$ 1d states (n_r)

$\ell = 0, \quad \ell_z = 0 \quad \longrightarrow$ 2s state ($\uparrow n_r$)

$$\boxed{n_r = n - \ell}$$

Arbitrary n

$$E_n = \omega(n + 3/2)$$

Number of states $= \dfrac{(n+1)(n+2)}{2} \quad \Big| \otimes 2$ (spin)

Shells

n	E	states	Number of states in the shell	Total number of states	parity
0	$\frac{3}{2}\omega$	1s	2	2	+
1	$\frac{5}{2}\omega$	1p	6	8	−
2	$\frac{7}{2}\omega$	1d, 2s	12	20	+
3	$\frac{9}{2}\omega$	1f, 2p	20	40	−
4	$\frac{11}{2}\omega$	1g, 2d, 3s	30	70	+

Magic numbers (closed shells) = 2, 8, 20, (70) ?

Closed shells in atoms result in noble gases (atoms with very high ionization potential):

atom	electron configuration	number of electrons	
He	$1s^2$	$Z = 2$	"Magic atoms"
Ne	$1s^2 2s^2 2p^6$	$Z = 10$	
Ar	$1s^2 2s^2 2p^6 3s^2 3p^6$	$Z = 18$	

Similarly magic nuclei are the nuclei with closed proton and neutron shells. They have maximum binding energy.

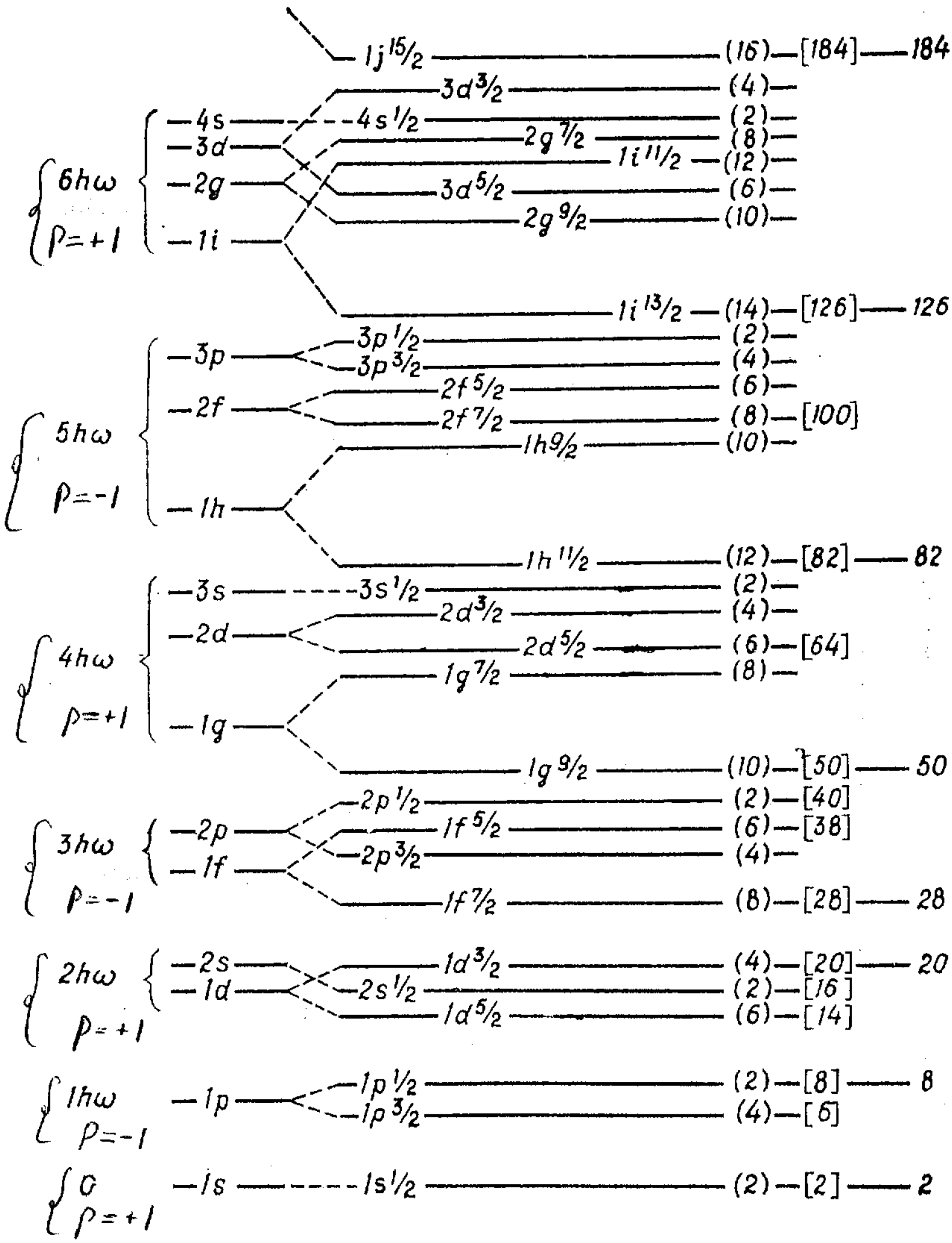
Magic nucleus	proton configuration	neutron configuration	Z	N
^{4}He (α-particle)	$1s^2$	$1s^2$	2	2
^{16}O	$1s^2 1p^6$	$1s^2 1p^6$	8	8
^{40}Ca	$1s^2 1p^6 2s^2 1d^{10}$	$1s^2 1p^6 2s^2 1d^{10}$	20	20

↑ n_r

Corrections to this simple picture are due to:

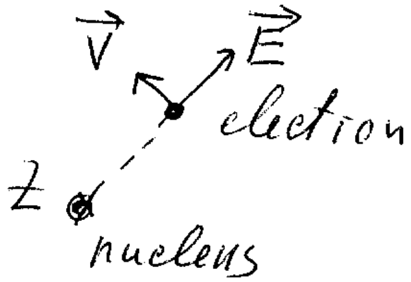
1) spin-orbit interaction

2) Coulomb proton-proton interaction



Single particle energy levels in spherical nuclei

Spin-orbit interaction in atoms



$$\vec E = \frac{Ze\vec r}{r^3}$$

Magnetic field in the intrinsic electron frame:

$$\vec{\mathcal{H}} = \frac{\vec V}{c}\times\vec E \simeq \frac{\vec V}{c}\times\frac{Ze\vec r}{r^3} = -\frac{Ze}{mcr^3}\underbrace{m\vec V\times\vec r}_{\vec\ell}$$

interaction:

$$H_{\ell s} = -\underset{2\mu_B\vec s}{\vec\mu}\,\vec{\mathcal{H}} = 2\frac{e\hbar}{2mc}\frac{Ze}{mcr^3}\vec s\cdot\vec\ell$$

$$\boxed{H_{\ell s} \sim \frac{Ze^2}{m^2c^2r^3}\vec\ell\cdot\vec s}\ , \quad \frac{1}{c^2}\text{-effect}$$

Strong interaction is velocity dependent therefore $\vec\ell\cdot\vec s$ interaction in nuclei is $1/c$-effect.

Effective $\vec\ell\cdot\vec s$ interaction in nuclei

$$H_{es} = \underbrace{\left(-\frac{20}{A^{2/3}}\right)}_{a}\vec\ell\cdot\vec s \quad \text{in MeV.}$$

$$H = H_0 + a\,\vec{l}\cdot\vec{s}$$

$$H_0 = \frac{p^2}{2m} + V(r)$$

$[H, \vec{l}\,] \neq 0 \Rightarrow \vec{l}$ is not conserved

$[H, \vec{l}^{\,2}] = 0 \Rightarrow \vec{l}^{\,2}$ is conserved

$[H, \vec{s}\,] \neq 0 \Rightarrow \vec{s}$ is not conserved

$[H, \vec{s}^{\,2}] = 0 \Rightarrow s^2 = 3/4$ is conserved

$$\vec{J} = \vec{l} + \vec{s}$$

$[H, \vec{J}\,] = 0 \Rightarrow \vec{J}$ is conserved

eigenstate of H $\quad |\psi\rangle = |l, s, j, j_z\rangle$

$$\bar{j} = \bar{l} + \bar{s} \Rightarrow j^2 = \vec{l}^{\,2} + 2\,\bar{l}\cdot\bar{s} + s^2 \Longrightarrow \langle\ \rangle \Rightarrow$$

$$\Rightarrow j(j+1) = l(l+1) + 2\langle \bar{l}\cdot\bar{s}\rangle + 3/4 \Rightarrow$$

$$\Rightarrow \langle\psi|\,\bar{l}\cdot\bar{s}\,|\psi\rangle = \frac{1}{2}\left[j(j+1) - l(l+1) - 3/4\right]$$

1) $j = l + 1/2$

$$\langle \vec{l}\vec{s} \rangle = \frac{(l+1/2)(l+3/2) - l(l+1) - 3/4}{2} = \frac{l}{2} \Rightarrow E_{l+\frac{1}{2}} = a\frac{l}{2}$$

2) $j = l - 1/2$

$$\langle \vec{l}\cdot\vec{s} \rangle = \frac{(l-1/2)(l+1/2) - l(l+1) - 3/4}{2} = -\frac{l+1}{2} \Rightarrow E_{l-\frac{1}{2}} = -a\frac{l+1}{2}$$

$$\Delta E_{ls} = E_{l+1/2} - E_{l-1/2} = a(l+1/2)$$

in nuclei

$a<0 \Rightarrow E_{l+1/2} < E_{l-1/2}$ negative spin orbit splitting

in atoms $a>0$

Oscillator shell model

$$E_{nlj} = \omega(n+1/2) + E_{ls}$$

energy levels are shown on page 59.

Magic numbers: 2, 8, 20, 50, 82, 126

In heavy nuclei because of the Coulomb repulsion number of neutrons ≃ ≃ $\frac{3}{2}$ number of protons

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Pairing of nucleons in nuclei

Strong interaction has tendency to pairing, therefore energy of the nucleus is minimum when total angular momentum is zero.

pp and nn pairing, there is no pn pairing.

Example: ^8Be $Z = N = 4$

proton configuration: $1s_{1/2}^2\, 1p_{3/2}^2$
neutron configuration: $1s_{1/2}^2\, 1p_{3/2}^2$

$j_1 = 3/2,\ j_2 = 3/2$

$J = j_1 + j_2 \longrightarrow \begin{cases} 3 \\ 2 \\ 1 \\ 0 \end{cases}$ 0 — ground state

An opposite example is the Hund's rule for atoms: spin is maximum.

Example: Carbon atom, C, $Z=6$

electron configuration: $1s^2 2s^2 2p^2$

the ground state has spin $S=1$.

The Hund's rule is a consequence of the Coulomb repulsion between electrons.

Back to nuclei: odd-even classification:

protons Z	neutrons N	angular momentum	
even	even	$J=0$	all neutrons as well as all protons are paired
even	odd	$J=j$ angular momentum of the external neutron	all protons are paired. Neutrons except the odd one are also paired
odd	even	$J=j$ angular momentum of the external proton	all neutrons are paired. Protons except the odd one are also paired
odd	odd	$\vec{J}=\vec{j_1}+\vec{j_2}$ can be arbitrary	one unpaired proton and one unpaired neutron

Pairing of nucleons is similar to the pairing of electrons in superconductors. Microscopic mechanisms are different. For nucleons the pairing is due to the properties of the strong interaction. For electrons it is due to the lattice vibrations.

Critical temperature for nuclear pairing: $T_c \sim 1\,\text{MeV} \sim 10^{10}\,K$.

"High temperature superconductivity".