# *Nuclear Moments*

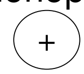


## *EM moments*

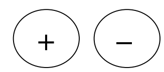
Electric charge → **E** field
Electric current → **B** field

Single charge $$E \propto \frac{1}{r^2}$$

Monopole or "Zeroth moment"

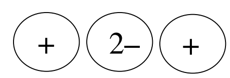



Dipole (First moment) $$E \propto \frac{1}{r^3}$$




Quadrupole (Second moment)

$$E \propto \frac{1}{r^4}$$






etc

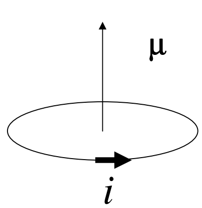
## *EM moments*

Similar for Magnetism (except for the monopole)

A spherical charge distribution → an electric monopole

A non-spherical charge distribution → an electric quadrupole

A circular current loop → a magnetic dipole




an 'Axial Vector'



## *Symmetry restrictions*

- Parity operator $\vec{r} \to -\vec{r}$

- Parity of moments =

$$\begin{cases} (-1)^L & Elec \\ (-1)^{L+1} & Mag \end{cases}$$

- L = 0 Monopole
- L = 1 Dipole
- L = 2 Quadrupole
- L = 3 Octupole
- etc

# Symmetry restrictions

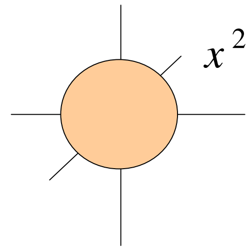
- 'All odd-parity multipole moments must vanish !!
- E1 (dipole), M2 (quadrupole), E3 (octupole) …..

<Moment> = $\int \psi^* \, Op \, \psi \; dv$



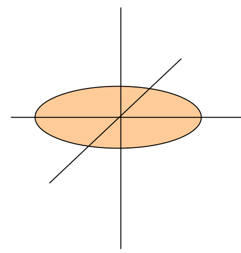
# Electric Quadrupole

- The operator $\left(3z^2 - r^2\right)$
is a measure of the deviation from spherical symmetry.



$$x^2 = y^2 = z^2 = \tfrac{1}{3} r^2$$
$$\therefore Q = 0$$

Spherical charge distribution



$$z^2 = 0 \therefore Q = -r^2$$

Planar charge distribution

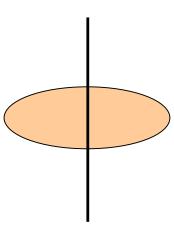


# Electric Quadrupole

- QM

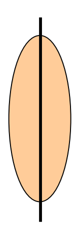
$$eQ = e \int \psi^* \left(3z^2 - r^2\right) \psi \; d\tau$$

- Spherical $Q = 0$
- Planar $Q = -\left\langle r^2 \right\rangle$
- Units of Q = $m^2$
- 1 barn (b) = $10^{-28}$ $m^2$



$Q < 0$

Oblate



$Q > 0$

Prolate



# Electric Quadrupole

$$\left.\begin{matrix} I = 0 \\ I = \tfrac{1}{2} \end{matrix}\right\} \rightarrow Q = 0$$

Nuclear energy states with $Q \neq 0$ can be split by the action of electric field gradients.

NQR, Mössbauer

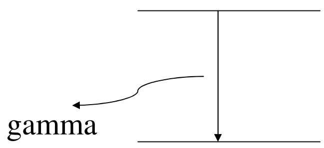
$$\left(3z^2 - r^2\right) \rightarrow \left(3\hat{I}_z^2 - \hat{I}^2\right)$$

Wigner-Eckart theorem

$^{176}$Lu has Q = +8.0 b

# *Electric Quadrupole*

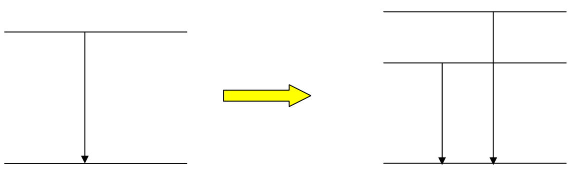
e.g. $^{57}$Fe Mössbauer Effect




$$I = \tfrac{3}{2} \rightarrow Q \neq 0$$

$$I = \tfrac{1}{2} \rightarrow Q = 0$$

If the nucleus is in an electric field gradient, the first excited state splits



$$I = \tfrac{3}{2} \rightarrow 2I + 1 = 4$$

Degeneracy of excited state

$$I_z = \pm \tfrac{3}{2} \quad \& \quad I_z = \pm \tfrac{1}{2}$$



# *Nuclear Models*



# *Nuclear models*

- Try to model the behaviour of the nucleus
- What kind of potential do the nucleons ‘feel’ ?
- Can we reproduce/predict the important nuclear parameters such as:
- Nuclear spin
- Nuclear magnetic moments
- Nuclear quadrupole moments
- Magic numbers



# *Liquid-drop model*

- Constant density
- B/A ~ constant (except for low-A)
- *cf* heat of vaporization /mass ~ constant for liquids
- Oldest and most “classical” model
- Semi-empirical mass formula
- Theory now gives a good account of the ‘volume’ term
- Pairing term suggests some sort of shell or energy level model

# *Liquid-drop model*

- Gives a good account of the average behaviour of nuclei
- Some nuclei are unusually stable
- **MAGIC NUMBERS** (particular values of Z and/or N)
- 2 8 20 28 50 82 126 (n)
- All EVEN
- *cf* inert ATOMS He, Ne, Kr, Ar etc
  "closed electron-shells"
- Segré chart Z = 20 has 6 stable isotopes whereas local average ~ 2
- Z = 50 has 10 stable isotopes whereas local average ~ 4
- Most of the known spontaneous neutron emitters have N = Magic + 1



# *Binding energy*

| A | N | Z | # stable nuclei |
|---|---|---|---|
| Even | Even | Even | 166 |
| Even | Odd | Odd | 8 |
| Odd | Even | Odd | 57 |
| Odd | Odd | Even | 53 |



# *Liquid-Drop model*

- Semi-empirical mass formula (von Weiszäcker 1935)
- Treat nucleus as a dense, incompressible, spherical liquid drop.

$\frac{B}{A}(MeV)$

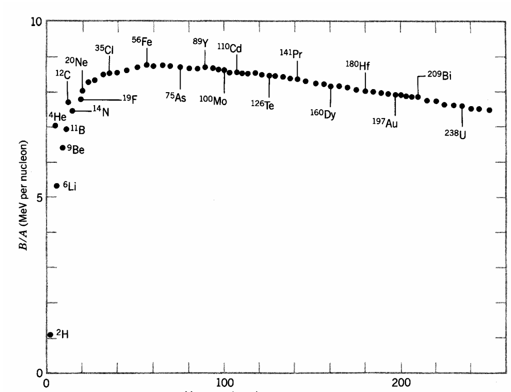



**Figure 3.16** The binding energy per nucleon.

$A$



Krane 3.16

# *Volume term*

- For $A \geq 30$

$$B/A \approx Const.$$

$$Vol \propto R^3 \therefore \propto A$$

$$B_V = a_V A$$

~ 8 MeV

A nucleon attracts its closest neighbours

## *Surface term*

- A nucleon near the surface has fewer neighbours
- As A ↑, the surface area ↑ so the number of 'surface nucleons' ↑
- Need to reduce Binding Energy

$$Area \propto R^2 \therefore \propto A^{2/3}$$

$$B_S = -a_S A^{2/3}$$



## *Coulomb repulsion term*

- Proton-proton repulsion makes nucleons less tightly bound.
- Long-distance Coulomb repulsion
- Need to reduce Binding Energy
- Z protons : how many p–p pairs ?

$$\frac{Z!}{2!(Z-2)!} = Z(Z-1)/2$$

$$Coulomb\ energy \propto \frac{1}{R} \therefore \propto A^{-1/3}$$

$$B_C = -a_C \frac{Z(Z-1)}{A^{1/3}}$$



## *Symmetry term*

- Light, stable nuclei tend to have Z ~ N ~ A/2.
- As A ↑ there is more p–p Coulomb repulsion so N > Z

Imbalance between N and Z

$$B_{Sym} = -a_{Sym} \frac{(A-2Z)^2}{A}$$

Reduces the effect as A ↑



## *Pairing term*

- Even–Even nuclei are particularly stable
- Z = even & N = even.
- Odd–Odd nuclei tend to be unstable
- [Later: Shell Model]

$$B_{pair} = \pm\delta$$

Even – Even → +
Odd – Odd → –
Even – Odd → 0

$$B_{pair} \propto A^{-3/4}$$

$$B_{pair} = a_{pair} A^{-3/4}$$

# *Energy terms*

- Best fit to experimental Binding Energy curve yields

$$
\begin{aligned}
a_V &= 15.5 MeV \\
a_S &= 16.8\ MeV \\
a_C &= 0.72\ MeV \\
a_{Sym} &= 23\ MeV \\
a_{pair} &= 34\ MeV
\end{aligned}
$$



# *Pairing term*

- We can see the effect of the pairing term in the Mass Parabolas

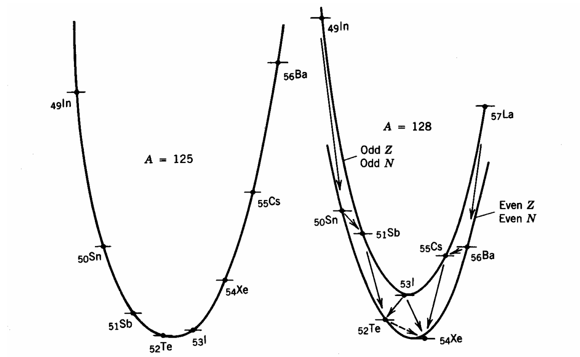



**Figure 3.18** Mass chains for $A$ = 125 and $A$ = 128. For $A$ = 125, note how the energy differences between neighboring isotopes increase as we go further from the stable member at the energy minimum. For $A$ = 128, note the effect of the pairing term; in particular, $^{128}$I can decay in either direction, and it is energetically possible for $^{128}$Te to decay directly to $^{128}$Xe by the process known as double $\beta$ decay.



Krane 3.18

# *Nuclear Models*

- Nuclear charge density is approx constant (from scattering experiments)

$$\rho(r) = \frac{\rho_o}{1 + \exp[(r - a)/b]}$$



# *Fermi-gas model*

- Analogous to the free-conduction-electron gas model of metals
- Each nucleon moves in an attractive NET potential (average of all interactions with neighbouring nucleons)
- 3D finite square-well

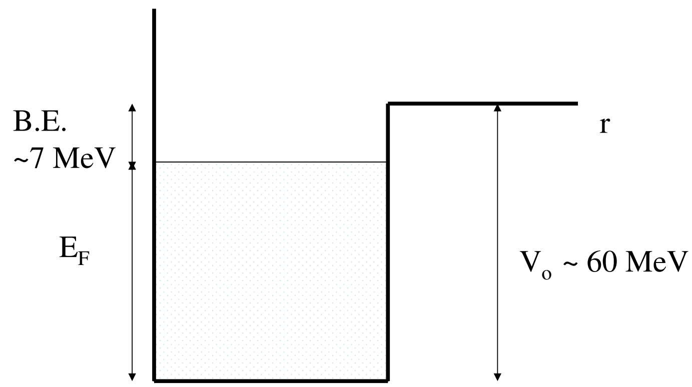
$$\rho(r) = \frac{\rho_o}{1 + \exp[(r - a)/b]}$$

# Fermi-gas model

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$

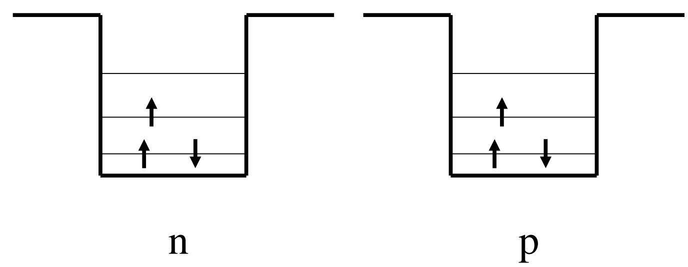
$$n = \frac{N}{V} = \frac{A}{\frac{4}{3}\pi r^3} = \frac{3}{4\pi R_o^3}$$

$$\therefore E_F = 8.5\times 10^{-12}\ J = 53\ MeV$$

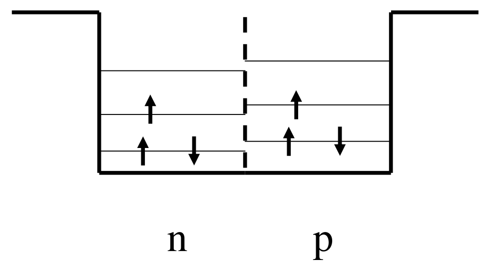





# Fermi-gas model




often use a composite diagram

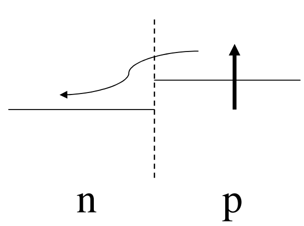



Z = N = 3; A = 6



# Fermi-gas model

The proton energy levels are slightly higher than those of the neutrons due to Coulomb repulsion




BETA stability: The nucleus will β-decay until nucleons fill the lowest energy levels

$p \rightarrow n$ OR $n \rightarrow p$



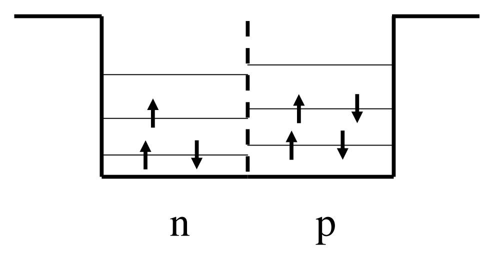
e.g. MIRROR nuclei ${}^{7}_{4}Be$ & ${}^{7}_{3}Li$

$N_1 = Z_2$

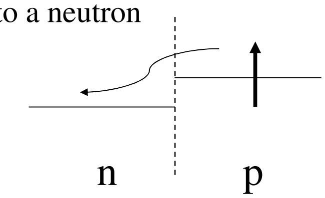
$N_2 = Z_1$

$A_1 = A_2$

both have spin-parity $\frac{3}{2}^-$



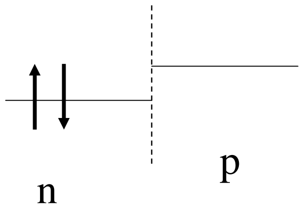

Convert one of Be protons to a neutron



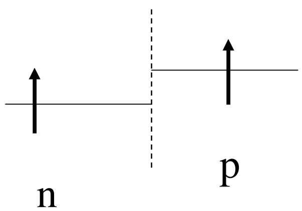

$${}^{7}_{4}Be + e^- \rightarrow {}^{7}_{3}Li + \nu_e$$

Electron-capture $t_{1/2} = 53.3d$

This model also explains the tendency to have even-even nuclei




preferable to






- Liquid-Drop model doesn't consider individual nucleons
- Doesn't describe

(1) ground and excited state spins and parities

(2) Magic numbers

(3) Magnetic moments

(4) Values of energy terms in SEMF

- Fermi-gas model suggests the validity of considering individual nucleons



# *Shell Model*

- Nucleons move independently in a net nuclear potential
- Form of the potential = ??
- Atomic case (H atom) is easy: electron moves in 'central' Coulomb potential created by an **external** agent i.e. the proton
- Nuclear case: the potential is created **internally**.



# *Nuclear Potential*

- Spherically-symmetric – no θ or ϕ dependence - easier
- Infinite square-well potential NO
- Simple-harmonic oscillator NO

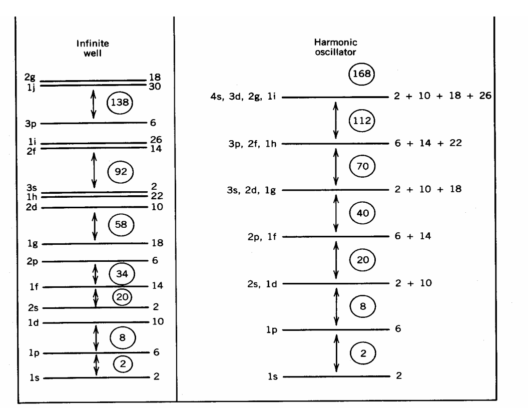



**Figure 5.4** Shell structure obtained with infinite well and harmonic oscillator potentials. The capacity of each level is indicated to its right. Large gaps occur between the levels, which we associate with closed shells. The circled numbers indicate the total number of nucleons at each shell closure.



Krane 5.4

- We've seen the form of the nuclear charge density

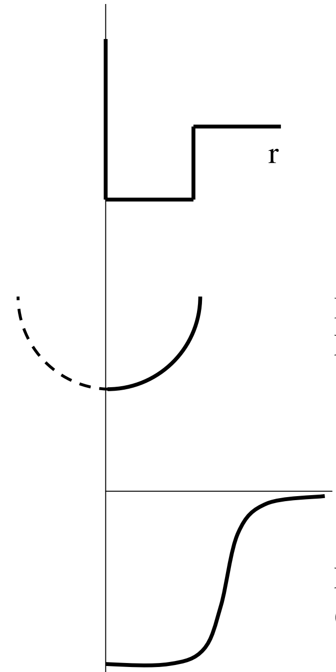
$$\rho(r)=\frac{\rho_o}{1+\exp[(r-a)/b]}$$

- Try a potential with the same r-dependence

$$V(r)=\frac{-V_o}{1+\exp[(r-R)/a_{skin}]}$$

Saxon-Woods potential






Infinite energy
Sharp edge

Infinite energy
Edge not sharp enough

Nuclear potential
(in-between)

$$V(r)=\frac{-V_o}{1+\exp[(r-R)/a_{skin}]}$$

$$V_0 \sim 50\ MeV$$

$$R=1.25A^{1/3}\ fm$$

$$a_{skin} \sim 0.52\ fm$$



## *Energy level scheme*

- s, p, d, f ….. $l=0, 1, 2.....$
(as in ATOM)
**BUT**

$$n=1, 2.....$$

is NOT the same as in ATOM !

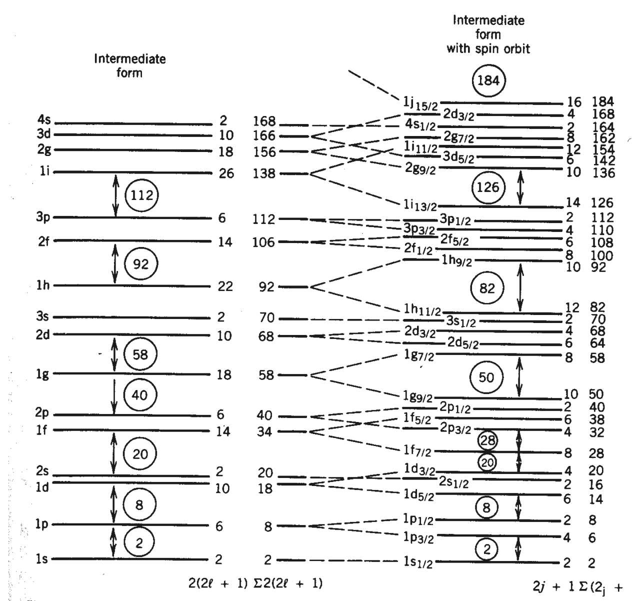
- An $l$ state has $2l+1$ values of $m_l$
- Total degeneracy = $2(2l+1)$
- e.g. $1s \longrightarrow l=0$

$m_l=0$ One level
Two nucleons
(± spin)

- 2 protons in 1s AND 2 neutrons in 1*s*
- p and n are both fermions but they're different particles !



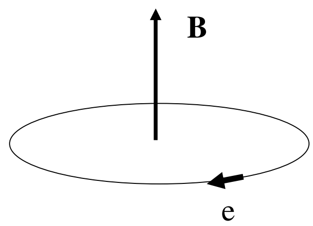
## *Shell Model*




Krane 5.6

# *Spin-orbit interaction*

**ATOM**




Orbital motion → magnetic field **B**
Electron's spin magnetic moment interacts with **B**
An *ELECTROMAGNETIC* interaction.

**NUCLEUS**

The spin-orbit interaction CANNOT be *ELECTROMAGNETIC* (not strong enough !).

It is a consequence of the nuclear *STRONG* force.



$$V_{so}(r)\,\vec{l}\cdot\vec{s}$$

$$\vec{j}=\vec{l}+\vec{s}$$

$$\vec{j}\cdot\vec{j}=\left(\vec{l}+\vec{s}\right)\cdot\left(\vec{l}+\vec{s}\right)$$

$$=\vec{l}\cdot\vec{l}+2\vec{l}\cdot\vec{s}+\vec{s}\cdot\vec{s}$$

$$\vec{l}\cdot\vec{s}=\tfrac{1}{2}\left[\vec{j}\cdot\vec{j}-\vec{l}\cdot\vec{l}-\vec{s}\cdot\vec{s}\right]$$

$$\left\langle\vec{l}\cdot\vec{s}\right\rangle=\tfrac{1}{2}\left[j(j+1)-l(l+1)-s(s+1)\right]\hbar^2$$

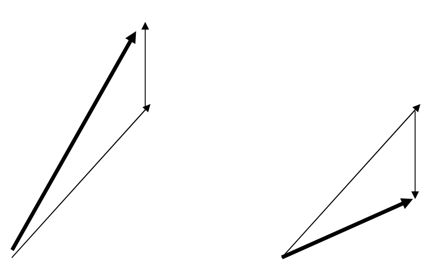


- A nucleon has $s=\frac{1}{2}$

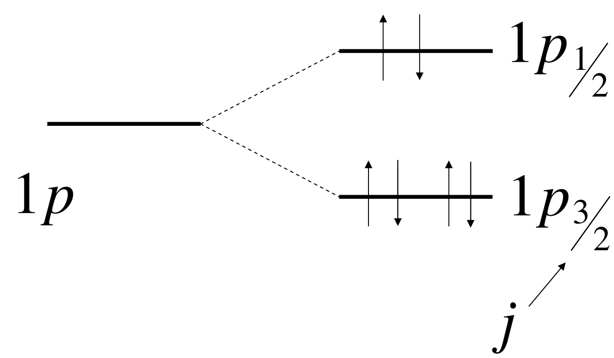
$$j=l\pm s=l\pm\tfrac{1}{2}$$

- e.g 1*p* state →

$$l=1\quad\therefore\quad j=\tfrac{1}{2},\tfrac{3}{2}$$








Degeneracies are given by the $2j+1$

values of $m_j$

Energy splitting

$$\left\langle\vec{l}\cdot\vec{s}\right\rangle_{j=l+1/2}-\left\langle\vec{l}\cdot\vec{s}\right\rangle_{j=l-1/2}$$

$$=\left(l+\tfrac{1}{2}\right)\hbar^2$$

## *Shell Model*

- Predicts Magic Numbers (occur at 'large' gaps in energy level scheme)
- Predicts ground state 'spins' of nuclei

$$^{208}_{82}Pb \qquad \begin{cases} Z = 82 \\ N = 126 \end{cases}$$

Both MAGIC

Completely filled sub-shells for both p and n

Exclusion Principle: spin and orbital angular momenta are 'paired'

$$\therefore \quad I = 0$$



## *Shell Model*

- Z or N = MAGIC ± 1
- Exclusion Principle: the extra nucleon (or 'hole') determines *I.*

| | $^{15}_{8}O$ | $^{16}_{8}O$ | $^{17}_{8}O$ |
|---|---|---|---|
| Z | 8 | 8 | 8 |
| N | 7 | 8 | 9 |
| Level | One n in 1p(1/2) | 1p (1/2) full | One n in 1d(5/2) |
| *I* | 1/2 | 0 | 5/2 |

"Doubly Magic"



## *Shell Model*

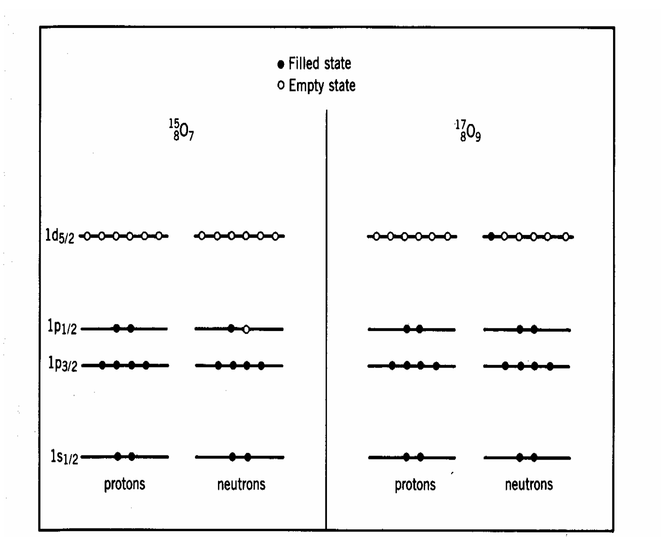



**Figure 5.7** The filling of shells in $^{15}O$ and $^{17}O$. The filled proton shells do not contribute to the structure; the properties of the ground state are determined primarily by the odd neutron.



Krane 5.7

## *Magnetic Moments Schmidt limits (1937)*

- Calculate the expected magnetic moment due to the odd (unpaired) nucleon

$$j = l + \tfrac{1}{2}$$
$$\langle \mu \rangle = \left[ g_l \left( j - \tfrac{1}{2} \right) + g_s / 2 \right]$$

$$j = l - \tfrac{1}{2}$$
$$\langle \mu \rangle = \left[ g_l \frac{j\left(j + \tfrac{3}{2}\right)}{(j+1)} - \frac{g_s}{2(j+1)} \right]$$

## *Schmidt limits*

$$g_l = \begin{cases} 1 & p \\ 0 & n \end{cases}$$

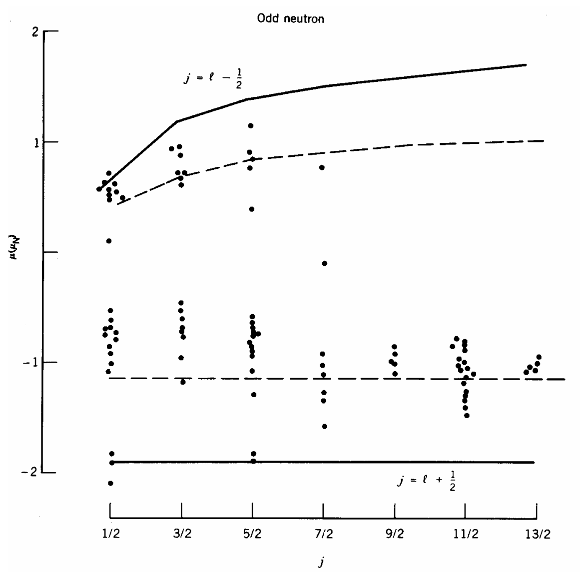
$$g_s = \begin{cases} +5.59 & p \\ -3.83 & n \end{cases}$$

One obvious flaw: model assumes that g-factors for nucleons in a nucleus are the same as the free-nucleon values above



## *Schmidt limits*

- Odd-neutron




ure 5.9 Experimental values for the magnetic moments of odd-neutron and d-proton shell-model nuclei. The Schmidt lines are shown as solid for $g_s$ = free) and dashed for $g_s = 0.6g_s$(free).



Krane 5.9a

## *Schmidt limits*

- Odd-proton

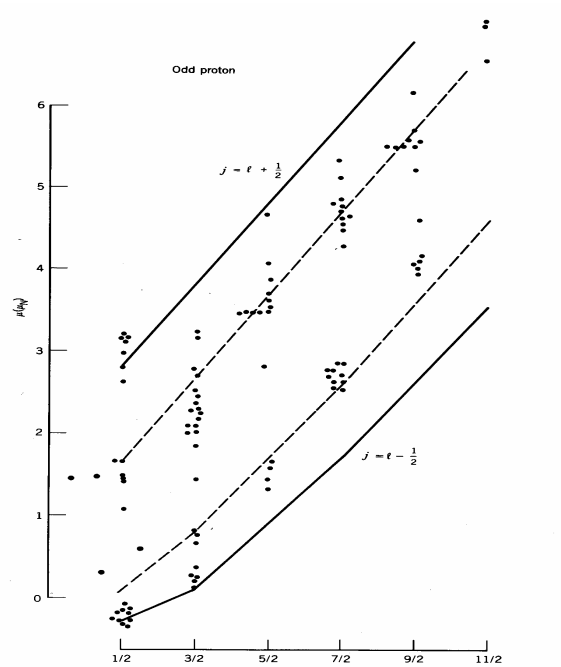


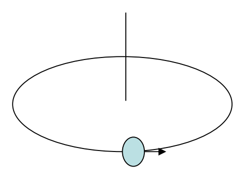

**Figure 5.9** Continued.



Krane 5.9b

## *Electric Quadrupole Moments*

- Nucleus with a single, unpaired proton



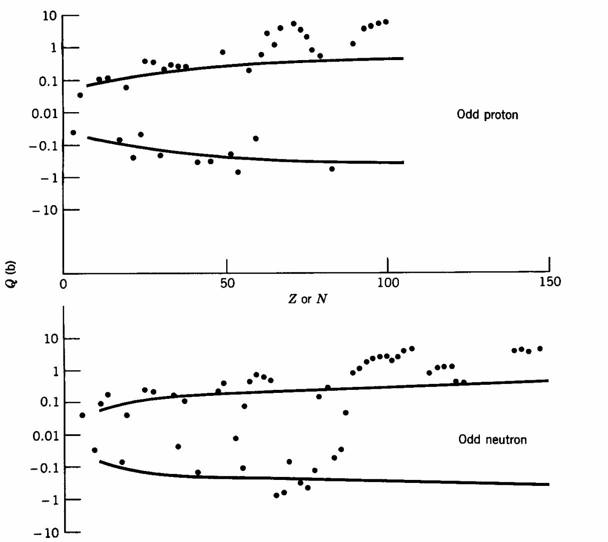
$$Q = -\langle r^2 \rangle$$

$$QM \rightarrow -\frac{2j-1}{2(j+1)}\langle r^2 \rangle$$

$$\langle r^2 \rangle = \tfrac{3}{5}R^2 = \tfrac{3}{5}R_o^2 A^{2/3}$$

- Theory gives sign but not magnitude (factor 2 – 3)

# *Electric Quadrupole Moments*




ʒure 5.10 Experimental values of electric quadrupole moments of odd-neutron d odd-proton nuclei. The solid lines show the limits $Q \sim \langle r^2 \rangle$ expected for ell-model nuclei. The data are within the limits, except for the regions $60 < Z < 80$, $> 90$, $90 < N < 120$, and $N > 140$, where the experimental values are more than order of magnitude larger than predicted by the shell model.



Krane 5.10

# *Collective Model*

- Shell Model assumes independent behaviour of nucleons
- Liquid-Drop Model assumes the opposite
- Collective Model takes features from both.
- Nucleons in unfilled subshells move independently in a net potential created by the filled 'core' nucleons (as in Shell Model)
- The potential is allowed to deform, as in a liquid. (The Shell Model assumes a static, spherically symmetric potential).
- "Valence nucleons"



# *Collective Model*

- Rainwater (1950): a single 'valence' nucleon outside the core interacts and deforms the core nuclear potential
- Every even-even nucleus has a first excited state $I = 2^+$

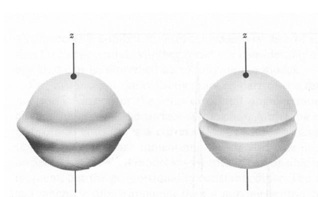


Figure 15-21 *Left:* Illustrating schematically an odd prot

Eisberg & Resnick 15.21



# *Vibrational states*

- Even-even nuclei
- A vibrating liquid drop.
- Average shape is spherical

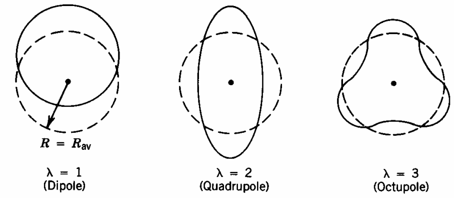



**Figure 5.18** The lowest three vibrational modes of a nucleus. The drawings represent a slice through the midplane. The dashed lines show the spherical equilibrium shape and the solid lines show an instantaneous view of the vibrating surface.

Krane 5.18

- Dipole mode violates conservation of momentum.
- Centre of mass is displaced – can't be the result of internal forces.

## *Vibrational states*

- Quadrupole mode: adds a 2nd-order spherical harmonic to the nuclear wavefunction
- 2 units of angular momentum (*l*)
- Phonon – a quantized vibration

$$\pi = (-1)^l = +1 \qquad \therefore\ 0^+ \rightarrow 2^+$$

- Two phonons result in a triplet

$$\therefore\ 0^+,\ 2^+, 4^+$$

- Energy = 2 x Energy of one-phonon $2^+$ state



## *Vibrational states*

- Octupole mode: adds a 3rd-order spherical harmonic to the nuclear wavefunction
- 3 units of angular momentum (*l*)

$$\pi = (-1)^3 = -1$$

$$\therefore\ 0^+ \rightarrow 3^-$$

$^{120}Te$

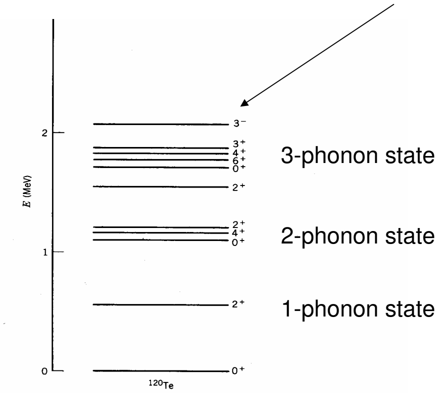



**Figure 5.19** The low-lying levels of $^{120}Te$. The single quadrupole phonon state (first $2^+$), the two-phonon triplet, and the three-phonon quintuplet are obviously seen. The $3^-$ state presumably is due to the octupole vibration. Above 2 MeV the structure becomes quite complicated, and no vibrational patterns can be seen.



Krane 5.19

## *Rotational states*

- Only observed in nuclei with non-spherical equilibrium shapes
- A range: 150 to 190; > 220
- Rotational inertia

$$K = \tfrac{1}{2}\Im\omega^2 \qquad l = \Im\omega$$

$$\therefore\ K = \frac{l^2}{2\Im}$$

$$l \rightarrow I\hbar$$

$$\therefore\ K = \frac{\hbar^2 I(I+1)}{2\Im}$$



## *Rotational states*

- The symmetry of even-even nuclei means that I = even

$$E(0^+) = 0$$

$$E(2^+) = 6\left(\frac{\hbar^2}{2\Im}\right)$$

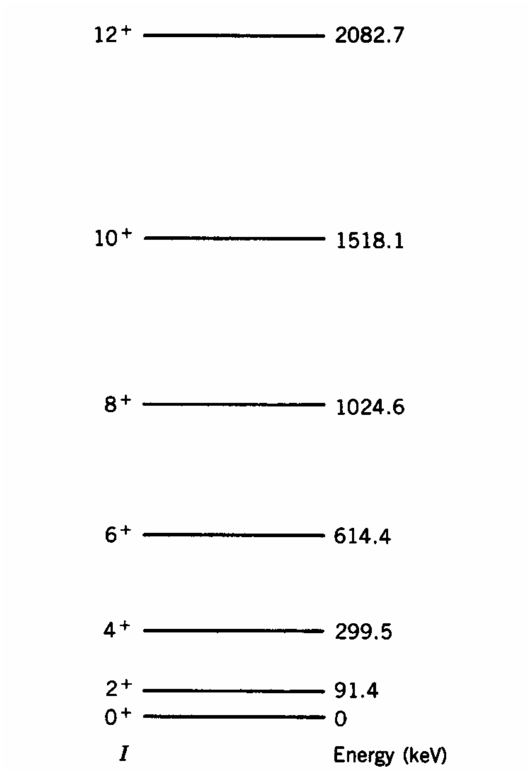
$$E(4^+) = 20\left(\frac{\hbar^2}{2\Im}\right)$$

$$E(6^+) = 42\left(\frac{\hbar^2}{2\Im}\right)$$

$$E(8^+) = 72\left(\frac{\hbar^2}{2\Im}\right)$$

etc

# *Rotational states in $^{164}Er$*






Krane 5.22