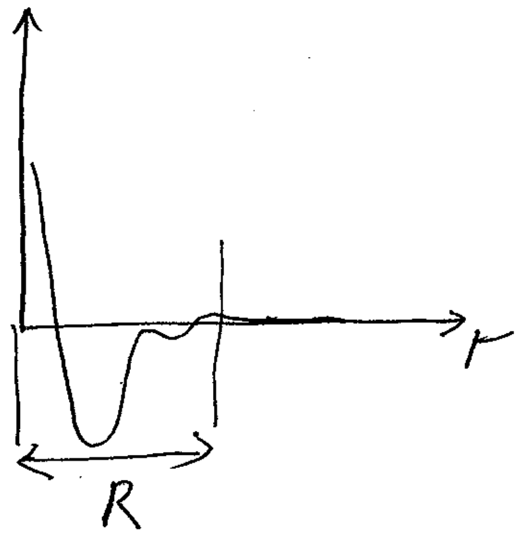
PART III

Scattering

1) Dominance of $l=0$ scattering at low energy.
2) Scattering phase
3) Scattering cross section and scattering amplitude
4) Scattering on a hardcore potential. Scattering length.
5) Scattering on a potential with shallow level. Virtual level.

Scattering

Consider motion of a particle with positive energy ($E>0$). Size of the potential is R.



Assume that

$$\boxed{E \ll \frac{\hbar^2}{mR^2}}$$

This is the low energy scattering regime.

For nuclei $R \sim 2\,fm = \frac{2}{197\,MeV} \Rightarrow$

$$\Rightarrow \frac{\hbar^2}{mR^2} = \frac{1}{940\left(\frac{2}{197}\right)^2} = 10\,MeV, \text{ so } \boxed{E \ll 10\,MeV}$$

Orbital angular momentum $L = mVR$

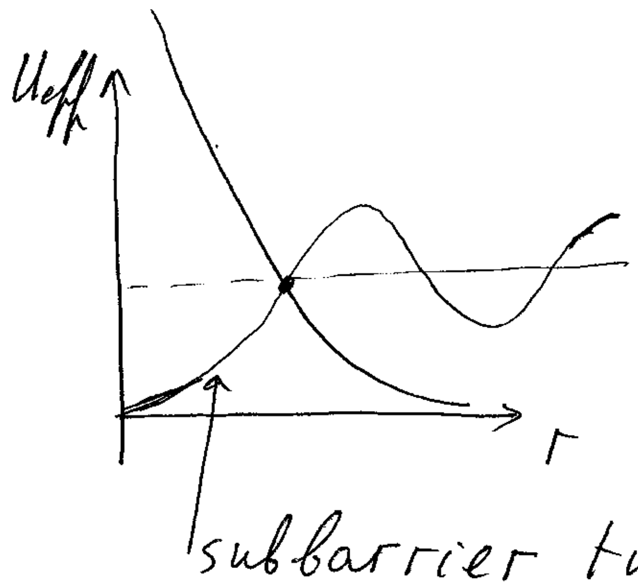
$$l = \frac{1}{\hbar}L = \frac{1}{\hbar}mVR = \sqrt{\frac{2\frac{mV^2}{2}}{\frac{\hbar^2}{mR^2}}} = \sqrt{\frac{2E}{\hbar^2/mR^2}} \ll 1$$

since l is quantized ($l=0,1,2,3..$) it means that $\boxed{l=0}$

Thus at low energy only s-wave scattering ($l=0$) is important.

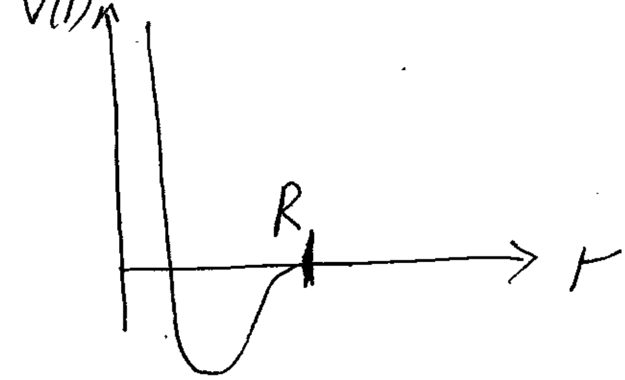
An alternative explanation of s-wave dominance: centrifugal barrier.

$$U_{eff}(r) = V(r) + \frac{L^2}{2mr^2} = V(r) + \frac{\hbar^2 l^2}{2mr^2}$$



- All the waves except $l=0$ are suppressed at small r because of the subbarrier
- tunneling. Hence they do not "feel" the potential $V(r)$.

Radial wave function $\chi(r)$ at $l=0$.



at $r<R$ $\chi(r)$ depends on $V(r)$

at $r>R$ $\chi(r) = A\sin(kr+\delta)$ – solution of the Schrodinger eq at $V=0$.

$\boxed{\delta \text{ is called the scattering phase}}$

The boundary condition

$$\left(\frac{\chi'}{\chi}\right)_{R-0} = \left(\frac{\chi'}{\chi}\right)_{R+0}$$ defines δ.

The scattering phase depends on the energy $\delta = \delta(E)$.

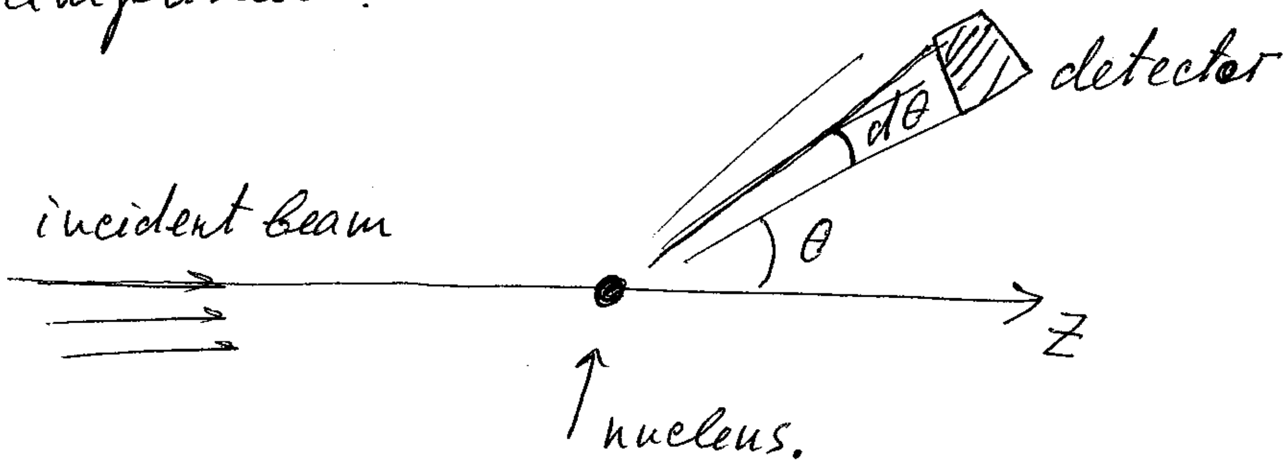
For free motion (no potential) $\delta = 0$.

<u>Theorem</u>: Potential $V(r)$ can be determined if scattering phase $\delta(E)$ is known for all energies.

Practically the theorem is not helpful.

How to measure δ ?

Scattering cross section and scattering amplitude.



wave function at $r > R$

$$\psi = e^{ikz} + \frac{f(\theta)}{r} e^{ikr}$$

incident wave — scattered wave

$$\boxed{k = p/\hbar \qquad v = \frac{p}{m}}$$

Number of particles detected per unit time = = Number of particles scattered to the solid angle $d\Omega = \sin\theta \, d\theta \, d\varphi$ per unit time.

$$dN_\Omega = |\psi_{scatt}|^2 v\, d\Omega\, r^2$$

$$\psi_{scatt} = \frac{f(\theta)}{r}$$

Hence $\quad dN_\Omega = v|f(\theta)|^2 d\Omega$

Note: dN_Ω is independent of r.

j = incident flux $\equiv$ number of incident particles per unit area per unit time.

$$j = |\psi_{incident}|^2 \cdot v = |e^{ikz}|^2 v = v$$

Definition of the differential cross section

$$\boxed{d\sigma = \frac{dN_\Omega}{j} = |f(\theta)|^2 d\Omega \quad \text{or} \quad \frac{d\sigma}{d\Omega} = |f(\theta)|^2}$$

Experimentally j is known and dN_Ω is known hence $\frac{d\sigma}{d\Omega}$ can be found from experiment.

Theoretically $f(\theta)$ can be expressed in terms of scattering phases δ_ℓ, $\ell = 0, 1, 2 \ldots$ – angular momentum.

$$\boxed{f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\cos\theta)}$$

this formula will be derived in Honours Quantum Mechanic

$$f_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right)$$

$P_\ell(\cos\theta)$ is the Legendre polinomial

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

. . . . . .

Thus measuring $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$ one can find all the scattering phases δ_ℓ.

At low energy

$\delta_\ell \sim (KR)^{2\ell+1}$; $\quad KR = \sqrt{\dfrac{P^2/m}{\hbar^2/mR^2}} \sim \sqrt{\dfrac{E}{\hbar^2/mR^2}} \ll 1.$

$\delta_0 \sim KR$ is small, but phases corresponding to higher ℓ are even smaller, say $\delta_1/\delta_0 \sim (KR)^2$. So all higher ℓ can be neglected, see also pp 35, 36.

Thus at small E

$$f(\theta) = \frac{1}{2iK}\left(e^{2i\delta_0} - 1\right) = \frac{1}{K} e^{i\delta_0} \frac{e^{i\delta_0} - e^{-i\delta_0}}{2i} =$$

$$= \frac{1}{K} e^{i\delta_0} \sin\delta_0$$

$\boxed{\delta_0 = -Ka \text{ , } a \text{ is called the scattering length}}$

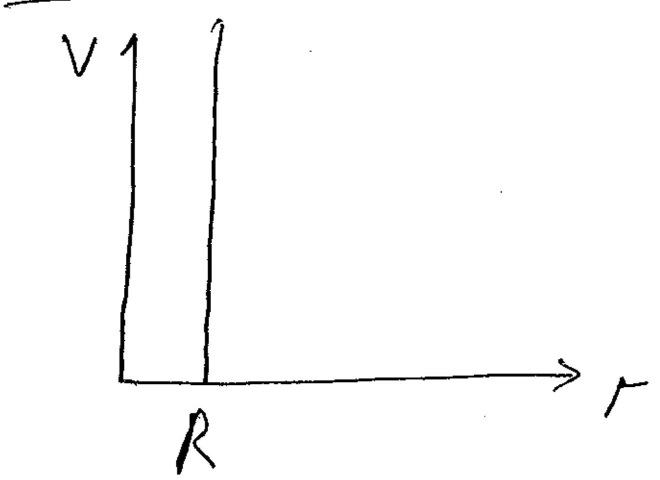
$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\sin^2\delta_0}{K^2} = a^2}$$

– independent of the angle because this is s-wave.

Total cross section

$$\boxed{\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = 4\pi a^2}$$

Example: scattering on the hard sphere



$$V(r) = \begin{cases} +\infty, & r < R \\ 0, & r > 0 \end{cases}$$

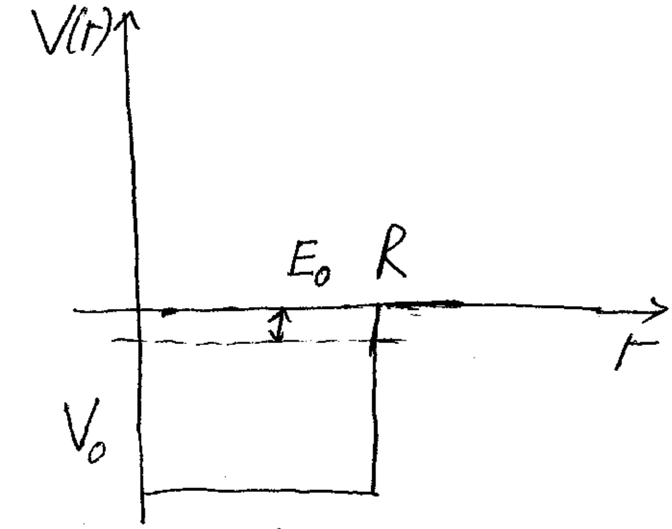
$$\chi = A \sin(kr + \delta)$$

Boundary condition $\chi(R) = 0 \Rightarrow \delta = -kR$.

The scattering length $\boxed{a = R}$

Total scattering cross section $\sigma = 4\pi R^2$ is four times larger than geometrical cross section πR^2

Example: Scattering on a potential with shallow bound state



The binding energy E_0 is small:

$$\boxed{E_0 \ll \frac{\hbar^2}{mR^2} \sim V_0}$$

see pp 10–12

Bound state

$\boxed{E = -E_0 < 0}$

$r < R$, $\chi(r) = A \sin k_1 r$

$r > R$ $\chi(r) = B e^{-\varkappa r}$

$$-\frac{\varkappa^2}{2\mu} = -E_0, \quad k_1 = \sqrt{2\mu(V_0 + E)} = \sqrt{2\mu(V_0 - E_0)}$$

$$\begin{cases} A \sin k_1 R = B e^{-\varkappa R} \\ A k_1 \cos k_1 R = -B \varkappa e^{-\varkappa R} \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{\tan k_1 R = -\frac{k_1}{\varkappa}}$$

Scattering

$\boxed{E > 0}$

$r < R$, $\chi(r) = A \sin k_2 r$

$r > R$, $\chi(r) = B \sin(kr + \delta)$

$$k = \sqrt{2\mu E}, \quad k_2 = \sqrt{2\mu(V_0 + E)}$$

$$\begin{cases} A \sin k_2 R = B \sin(kR + \delta) \\ A k_2 \cos k_2 R = B k \cos(kR + \delta) \end{cases}$$

$$\Rightarrow \boxed{\tan k_2 R = \frac{k_2}{k} \tan(kR + \delta)}$$

Shallow level $\Rightarrow E_0, E \ll V_0$

$$\Rightarrow \left.\begin{array}{l} K_1 = \sqrt{2\mu(V_0 - E_0)} \approx \sqrt{2\mu V_0} \\ K_2 = \sqrt{2\mu(V_0 + E)} \approx \sqrt{2\mu V_0} \end{array}\right\} \Rightarrow K_1 \approx K_2 \Rightarrow$$

$$\Rightarrow \tan K_1 R \approx \tan K_2 R \Rightarrow -\frac{K_1}{\varkappa} = \frac{K_2}{K} \tan(KR+\delta)$$

$$\Rightarrow \boxed{\tan(KR+\delta) = -\frac{K}{\varkappa}}$$

we will see that $KR \ll \delta$, hence

$$\downarrow$$

$$\boxed{\tan\delta = -\frac{K}{\varkappa}}$$

1) "Very" low energy, $K \ll \varkappa$

$\delta \approx -\frac{K}{\varkappa}$, hence the scattering length

$a = 1/\varkappa$, $\quad \sigma = 4\pi a^2 = \frac{4\pi}{\varkappa^2} = \frac{4\pi}{2m E_0}$

2) $K \sim \varkappa \ll \frac{1}{R}$

$$\cos\delta = \frac{\varkappa}{\sqrt{\varkappa^2+K^2}}, \quad \sin\delta = -\frac{K}{\sqrt{\varkappa^2+K^2}}$$

$$f_0 = \frac{1}{K} e^{i\delta} \sin\delta = -\frac{1}{K} \frac{(\varkappa - iK)}{\sqrt{\varkappa^2+K^2}} \frac{K}{\sqrt{\varkappa^2+K^2}} = -\frac{1}{\varkappa + iK}$$

↑ see page 41

$$\sigma = 4\pi |f_0|^2 = \frac{4\pi}{\varkappa^2 + k^2} = \frac{4\pi}{2m\left(\frac{\varkappa^2}{2m} + \frac{k^2}{2m}\right)} = \frac{4\pi/2m}{E + E_0}$$

resonance formula

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pn scattering

$S = 1$ triplet $\uparrow_n \uparrow_p$ the same channel as deutron.

Theoretical prediction (see page 44):

$$a_t = \frac{1}{\varkappa} = \frac{1}{\sqrt{2mE_0}} \longrightarrow \frac{1}{\sqrt{m_p E_0}} = \frac{1}{\sqrt{940 \cdot 2.2}} = \frac{1}{45.5\,\text{MeV}} =$$

$$m \longrightarrow \mu = m_p/2$$

$$= 4.3\,\text{fm} \overset{?}{>>} R \sim 2\,\text{fm}$$

Experiment $a_t = 5.35\,\text{fm}$

reasonable agreement

$S = 0$ singlet channel: $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_p |\downarrow\rangle_n - |\downarrow\rangle_p |\uparrow\rangle_n\right)$

$a_s = -23.55\,\text{fm}$ - experiment

$a_s < 0 \Rightarrow$ there is no bound state
(see page 44).

<u>Question:</u> why a_s is so big ($a_s >> R$)?

This is called virtual level, there is no bound state but if one just slightly increases the attractive potential the bound state would appear.

Generalized Fermi statistics (see page 32,

$-1 = (-1)^{\ell}(-1)^{S+1}(-1)^{T+1}$, <u>$\ell = 0$</u> because we consider low energy scattering.

Hence $\boxed{(-1)^{S+1}(-1)^{T+1} = -1}$

$S=1 \Rightarrow T=0$ this is the deutron channel.

$S=0 \Rightarrow T=1$, hence the virtual level has isospin $\boxed{T=1}$. and therefore it also must manifest itself in pp ($T_z=1$) and nn ($T_z=-1$) scattering

For pp and nn scattering only singlet ($S=0$) scattering is possible. This follows from the generalized Fermi statistics and the same follows from the usual Fermi statistics

Experiment $\quad pp: \quad a_s = -7.82\ fm$

$\qquad\qquad\quad nn: \quad a_s = -16.6\ fm$

Values of a_s are different because of the Coulomb repulsion for pp. If the repulsion is taken away then

$$a_s = -7.82\ fm \to -17.1 \pm 0.2\ fm$$

for protons agrees with that for neutrons. This confirms charge independence of strong interaction. The value $a_s \approx -17\ fm$ is still slightly different from that for pn, $a_s \approx -23\ fm$. This is a small violation of the isotopic symmetry.