A/ Prof. C. Ulrich

Electromagnetism PHYS2050

2 Electrostatics

2.5 Work and Energy of the electric Field

2.5.1 Definition of the electrostatic Energy

The energy of the electric field can be calculated from the force which acts on one single point charge: $\vec{F}(\vec{r}) = Q_0 \cdot \vec{E}(\vec{r})$:

How much 'work' is required to move a charge from a to b?

Advantage of the potential: the integration is independent of the path of the integration. Only the start end the end point are required.

$$W \;=\; \int_a^b \vec{F}(\vec{r})\; d\vec{l} \;=\; -Q_0 \int_a^b \vec{E}(\vec{r})\; d\vec{l} \;=\; Q_0\; [\phi(b) - \phi(a)]$$

Units:

$$1 \text{ Joule} \;=\; 1\; N \cdot m \;=\; 1\; kg\, \frac{m}{s^2}\; m \;=\; As \cdot V \;=\; V \cdot A \cdot s$$

The potential difference is equal to the work per unit charge: $[\phi(b) - \phi(a)] = W/Q_0$

Work which is required to bring a charge from infinity to a certain point:

$$W \;=\; Q_0\; [\phi(\vec{r}) - \phi(\infty)] \;=\; Q_0 \cdot \phi(\vec{r})$$

In this case we did set the reference point to infinity. Note: $\vec{E}(\vec{r}) \propto 1/|\vec{r}|^2$.

2.5.2 Energy of a distribution of point charges

Let us consider a collection of charges.

The charges are either positive of negative. Thus, there are either attractive or repulsive forces between the individual charges.

The entire collection of charges contains in total a certain amount of energy.

1. There is no work required to place the first charge q_1 since there is initially no electric field.
2. In order to place the second charge q_2 the required work is: $W = q_2\; \phi_1(\vec{r}_2)$

$$W_2 \;=\; \frac{1}{4\pi\varepsilon_0}\; q_2\; \left(\frac{q_1}{|\vec{r}_2 - \vec{r}_1|} \right)$$

3. Now we bring a third charge q_3 to the ensemble of the first two charges:

$$W_3 \;=\; \frac{1}{4\pi\varepsilon_0}\, q_3 \left(\frac{q_1}{|\vec{r}_3-\vec{r}_1|}+\frac{q_2}{|\vec{r}_3-\vec{r}_2|}\right)$$

4. For the fourth charge q_4 we obtain:

$$W_4 \;=\; \frac{1}{4\pi\varepsilon_0}\, q_4 \left(\frac{q_1}{|\vec{r}_4-\vec{r}_1|}+\frac{q_2}{|\vec{r}_4-\vec{r}_2|}+\frac{q_3}{|\vec{r}_4-\vec{r}_3|}\right)$$

The total required charge for the entire ensemble of charges is:

$$\begin{aligned} W &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{|\vec{r}_2-\vec{r}_1|}+\frac{q_1q_3}{|\vec{r}_3-\vec{r}_1|}+\frac{q_1q_4}{|\vec{r}_4-\vec{r}_1|}+\frac{q_2q_3}{|\vec{r}_3-\vec{r}_2|}+\frac{q_2q_4}{|\vec{r}_4-\vec{r}_2|}+\frac{q_3q_4}{|\vec{r}_4-\vec{r}_3|}\right) \\ &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_iq_j}{|\vec{r}_j-\vec{r}_i|} \end{aligned}$$

This expression can be simplified:

$$\begin{aligned} W &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_iq_j}{|\vec{r}_j-\vec{r}_i|} \\ &= \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_iq_j}{|\vec{r}_j-\vec{r}_i|} \\ &= \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\,\frac{q_j}{|\vec{r}_j-\vec{r}_i|}\right) \;=\; \frac{1}{2}\sum_{i=1}^{n} q_i\,\phi(\vec{r}_i) \end{aligned}$$

This is the amount of energy which it takes at assemble the charge distribution. We gain this energy if we disassemble the charge distribution again.

2.5.3 Energy of a continuous charge distribution

Let us consider a continuous charge distribution. The energy of the charge distribution is:

$$W \;=\; \frac{1}{2}\sum_{i=1}^{n} q_i\,\phi(\vec{r}_i) \;=\; \frac{1}{2}\int\limits_V \varrho(\vec{r}\,')\,\phi(\vec{r}\,')\,d\vec{r}\,'$$

Using the Gauss' law ($\mathrm{div}\vec{E}=\varrho/\varepsilon_0$) we obtain:

$$W \;=\; \frac{\varepsilon_0}{2}\int\limits_V (\nabla\cdot\vec{E}(\vec{r}\,'))\,\phi(\vec{r}\,')\,d\vec{r}\,'$$

We can now apply the rules for the divergence (see chapter 1).

$$\int\limits_L f(\nabla\vec{A})\,d\vec{l} \;=\; -\int\limits_L \vec{A}\cdot(\nabla f)\,d\vec{l} + \oint\limits_S f\cdot\vec{A}\,d\vec{a}$$

Using this rule we obtain:

$$\begin{aligned} W &= \frac{\varepsilon_0}{2}\left(-\int\limits_V \vec{E}(\vec{r}')\cdot(\nabla\phi(\vec{r}'))\ d\vec{r}' + \oint\limits_S \phi(\vec{r}')\cdot\vec{E}(\vec{r}')\ d\vec{a}\right) \\ &= \frac{\varepsilon_0}{2}\left(\int\limits_V \left(\vec{E}(\vec{r}')\right)^2\ d\vec{r}' + \oint\limits_S \phi(\vec{r}')\cdot\vec{E}(\vec{r}')\ d\vec{a}\right) \end{aligned}$$

The entire expression for the work contains a volume integral and a surface integral over the lines of the electric field which leave through the surface around the volume.

Let us assume that we integrate over the entire volume, i.e. from $-\infty$ to ∞. The surface is located in infinity and the area integral goes to zero. Since $\phi \propto 1/|r|$ and $\vec{E} = -\nabla\phi \propto 1/|r|^2$ the integrand is $\propto 1/|r|^3$. The area is $\propto |r|^2$. Therefore, the entire surface integral is $\propto 1/|r|$ and must vanish for large distances.

$$\boxed{W = \frac{\varepsilon_0}{2}\int\limits_V \left(\vec{E}(\vec{r}')\right)^2\ d\vec{r}'}$$

In the case of two charge distributions the energy of the electric field is:

$$\boxed{W = \frac{\varepsilon_0}{2}\int\limits_V\int\limits_{V'} \frac{\varrho(\vec{r}')\varrho(\vec{r})}{|\vec{r}-\vec{r}'|}\ d\vec{r}\ d\vec{r}'}$$

2.6 The Method of Images

A charge in front of a conductor!

2.6.1 Conductors

- In an **insulator** the electrons are localized at the position of the atoms. They cannot move freely.
- In a **conductor** (metals like: Cu, Fe, Au, Al, ...) some of the electrons are free, the conduction electrons, and can move through the entire material. They still experience a 'resistance', i.e. a finite electrical conductivity.
  Exception: superconductors like Pb ($T_c = 7.2\ K$), $YBa_2Cu_3O_7$ ($T_c = 92\ K$), or $HgBa_2Ca_3Cu_4O_{10}$ ($T_c = 124\ K$, $T_c = 160\ K$ at a pressure of $10\ GPa$).

(i) **Inside a conductor: $\vec{E}(\vec{r}) = 0$**

If there is any electric field inside the conductor, the electrons would move in order to compensate the electric field.

(ii) **Inside a conductor: $\varrho(\vec{r}) = 0$**

From the Gauss law (Maxwell law) $\mathrm{div}\vec{E}(\vec{r}) = \varrho(\vec{r})/\varepsilon_0$.
If the electric field is zero, the charge density $\varrho(\vec{r})$ must be zero as well.
Question: Where does the electric charge of a conductor go to in case of an external electric field $\vec{E}_{\text{ext.}}$?

(iii) **Any net charge resides on the surface of the conductor.**

In an external electric field $\vec{E}_{\text{ext}}$ the negative charges move against an external field $\vec{E}_{\text{ext}}$ and the positive charges move with the external field $\vec{E}_{\text{ext}}$. At the edge of the conductor the charges will pile up.
These **induced charges** produce an electric field $\vec{E}_{\text{ind}}$ which acts against the external electric field $\vec{E}_{\text{ext}}$. As many charges move inside the conductor until the internally induced electric field $\vec{E}_{\text{ind}}$ cancels out the external electric field $\vec{E}_{\text{ext}}$.

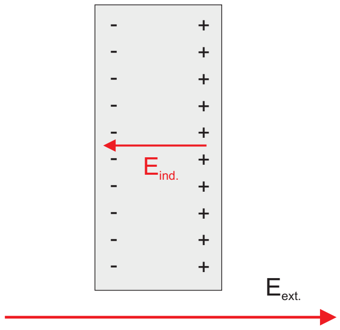

Figure 2.12: The surface charges create an internal induced electric field $\vec{E}_{\text{ind.}}$ which compensates the external electric field $\vec{E}_{\text{ext}}$.

(iv) **A conductor is a equipotential.**

Consider two points a and b:

$$\phi(b) - \phi(a) = -\int \vec{E}\, d\vec{l} = 0 \qquad \longrightarrow \qquad \phi(a) = \phi(b)$$

(v) **Outside the conductor: $\vec{E}(\vec{r})$ is perpendicular to the surface.**

Any tangential component of the electric field is immediately destroyed by the freely moving charges, which try to compensate any tangential component.

(vi) **Energy of the electric field of a conductor:**

If the charges are located at the surface of a conducting sphere, the energy is: $W = \frac{1}{8\pi\varepsilon_0}\,\frac{q^2}{R}$.
If the charges would be homogeneously distributed throughout the conducting sphere, the energy would be: $W = \frac{3}{20\pi\varepsilon_0}\,\frac{q^2}{R}$.
This is slightly larger than the energy for the charges located at the surface. Thus, this consideration about the energy also proves that the charges must be located at the surface of the sphere.

Induced Surface Charges

The external electric field induces a charge distribution on the surface of the conductor.

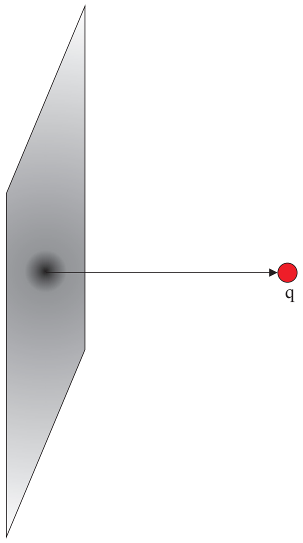

Figure 2.13: Induced charge distribution on the surface of a conductor. Example: point charge in front of a conducting plate.

Force on a conductor

The electric field outside a conductor is:

$$\vec{E} \quad = \quad \frac{\sigma}{\varepsilon_0}\,\vec{n}_0$$

where σ is the surface charge:

$$\sigma \quad = \quad -\varepsilon_0\,\frac{\partial\phi}{\partial n}$$

Force: $\vec{F} = \sigma \cdot \vec{E}_{\text{ext.}}$?

Problem: the electric field is discontinuous at the surface. Therefore, the average of the electric field at the surface has to be taken, i.e. for the electric field inside and outside the conductor:

$$\vec{F} \quad = \quad \sigma \cdot \vec{E}_{\text{av.}} \quad = \quad \frac{1}{2}\sigma\left(\vec{E}_{\text{inside}} + \vec{E}_{\text{outside}}\right)$$

Note that the electric field inside the conductor is zero. Therefore, the force per unit area is only arising from $\vec{E}_{\text{outside}}$, i.e. half of the entire force:

$$\vec{F} \quad = \quad \frac{1}{2\varepsilon_0}\,\sigma^2 \cdot \hat{n}$$

2.6.2 The Classical Image Problem

A point charge is located at a distance d from a conducting plate:

Question: Is the potential:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r}|} ?$$

Answer: NO! Since q induces a negative charge at the surface!

Question: How does this charge-distribution look like?

Solution:

Solve the Poisson equation in the region $z > 0$ with a single point charge q at $(0, 0, d)$ and with the proper boundary conditions:

1. $\phi = 0$ when $z = 0$
2. $\phi \rightarrow 0$ for q at large distance from the plate

$\phi = 0$ at the surface of the metallic plate holds if the metallic plate is grounded or in case of an infinite large conducting plate.

Important Trick: Mirror Images!

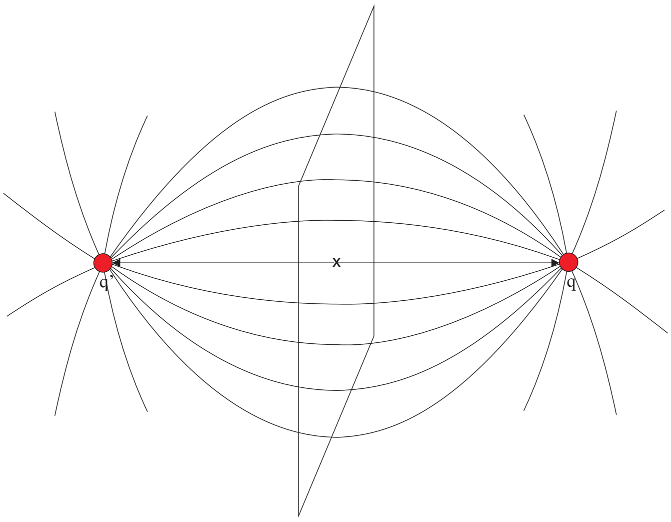

Figure 2.14: Mirror image of a charge in front of a metallic plate.

As a model we can place a 'mirror charge' on the other side of the conducting plate in such a way that the same boundary conditions apply:

1. $\phi = 0$ when $z = 0$ i.e. between the two charges
2. $\phi \rightarrow 0$ at a large distance

The potential of these two charges, the charge q at d and the mirror charge $q' = -q$ at $-d$, can be written as:

$$\phi(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

Induced Surface Charge

σ is the surface charge:

$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial n}$$

and $\frac{\partial\phi}{\partial n}$ is the normal derivative at the surface ($z = 0$).

$$\frac{\partial\phi(x, y, z)}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q(z-d)}{(x^2 + y^2 + (z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2 + y^2 + (z+d)^2)^{3/2}}\right]$$

we obtain:

$$\sigma(x, y) = \frac{1}{2\pi}\,\frac{-qd}{(x^2 + y^2 + d^2)^{3/2}}$$

The surface charge is negative (as it should be) and largest directly below the point charge (as expected).

Total induced charge: Integral over the area of the plate!

$$Q = \int\limits_A \sigma \, d\vec{a}$$

$$Q = \int\limits_0^{2\pi}\int\limits_0^{\infty} \frac{1}{2\pi}\,\frac{-qd}{(\underbrace{x^2 + y^2}_{r^2} + d^2)^{3/2}}\, r\, dr\, d\phi$$

$$= \int\limits_0^{2\pi} \frac{1}{2\pi}\,\frac{qd}{\sqrt{r^2 + d^2}}\Bigg|_0^{\infty} d\phi = \frac{qd}{\sqrt{r^2 + d^2}}\Bigg|_0^{\infty} = -q$$

as expected.

Force:

The charge is attracted towards the conducting plate!

Again: Trick with the two mirror charges ($-q$ at a distance of 2 d).

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{(2d)^2}\,\vec{e}_z$$

Energy:

Energy of two point charges:

$$W = -\frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{2d}$$

But: The stored energy of a point charge in front of a conducting plate is only 1/2 of this energy since the electric field behind the metallic plate is zero.

$$W = -\frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4d}$$

Energy of the electric field: $W = \frac{\varepsilon_0}{2} \int E(\vec{r})^2 \, d\vec{r}$

The since the surface charges compensate the external electric field created by the point charge, the electric field exists only outside the conducting plate.

Inside the conducting plate the electric field is zero.

$$\begin{aligned} W &= \int_{\infty}^{d} F(\vec{r})\, d\vec{l} = \frac{1}{4\pi\varepsilon_0} \int_{\infty}^{d} \frac{q^2}{4z^2}\, dz \\ &= \frac{1}{4\pi\varepsilon_0} \left(-\frac{q^2}{4z}\right)\Bigg|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\,\frac{q^2}{4d} \end{aligned}$$

2.6.3 Potential of a point charge near a conducting sphere

Let us consider the induced charge in case of a point charge close to a grounded conducting sphere.

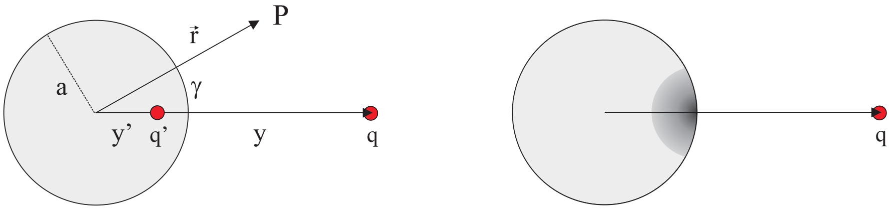

Figure 2.15: Point charge placed in front of a conducting sphere.

The external point charge q is located at y. We used the concept of a mirror charge q', which is located at $\vec{y}'$. The potential can be written as:

$$\begin{aligned} \phi((\vec{r})) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{r}-\vec{y}|} + \frac{\vec{q}'}{|\vec{r}-\vec{y}'|}\right) \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{a\sqrt{1+(\frac{y}{a})^2 - 2\frac{y}{a}\cos\gamma}} + \frac{q'}{y'\sqrt{1+(\frac{a}{y'})^2 - 2\frac{a}{y'}\cos\gamma}}\right) \end{aligned}$$

In the next step we have to determine q' and y'.

Therefore, we use the boundary conditions, i.e. $\phi = 0$ at the surface of the conducting sphere $r = a$. In the model of the mirror charge this condition must be fulfilled at $r = a$ between the charge and the mirror charge on the y-axis, i.e. at an angle of $\gamma = 0$:

$$\frac{q}{a\sqrt{1+(\frac{y}{a})^2 - 2\frac{y}{a}\cos\gamma}} + \frac{q'}{y'\sqrt{1+(\frac{a}{y'})^2 - 2\frac{a}{y'}\cos\gamma}} = 0$$

$$qy' \sqrt{1 - 2\frac{a}{y'} + \left(\frac{a}{y'}\right)^2} = -q'a \sqrt{1 - 2\frac{y}{a} + \left(\frac{y}{a}\right)^2}$$

This condition is fulfilled when:

$$\frac{q}{a} = -\frac{q'}{y'} \qquad \text{and} \qquad \frac{y}{a} = +\frac{a}{y'}$$

From this expression we can derive the size and position of the mirror charge q':

$$q' = -\frac{a}{y}\, q \qquad \text{and} \qquad y' = \frac{a^2}{y}$$

A consequence of this result is that the mirror charge goes to zero if the distance y of the charge from the sphere goes to infinity. In this case, the location of the mirror charge converges towards the center of the sphere.

If the charge is placed very close to the sphere ($y \approx a$), the mirror charge is approximately equal but opposite to q ($q' = -q$) and the location of the mirror charge is close to the surface, but inside the conducting sphere.

Finally the potential of a point charge in front of a grounded conducting sphere can be written as:

$$\phi(\vec{(r)}) = \frac{1}{4\pi\varepsilon_0}\, q \left(\frac{1}{|\vec{r} - \vec{y}|} - \frac{a}{y|\frac{a^2}{y^2}\vec{y} - \vec{r}|} \right)$$

This solution is valid for the space outside the sphere. Inside the conducting sphere the electric field is zero.

Analogously to the calculation of the surface charge density in case of an infinite conducting plate, we can determine the surface charge density on a grounded conducting sphere. Note that in this case spherical coordinates must be used:

$$\sigma(r = a) = -\frac{q}{4\pi a y} \frac{\left(1 - \frac{a^2}{y^2}\right)}{\left(1 + \frac{a^2}{y^2} - \frac{2a}{y}\cos\gamma\right)^{3/2}}$$

In case of a non-grounded conducting sphere the charges redistribute on the surface changes in such a way, that a similar mirror charge is created opposite to the external point charge (see equation above). The remaining charge on the surface $(Q - q')$ is uniformly distributed and must be taken into account when the energy or the force, which acts on the external charge, is calculated.

2.7 The Capacitor

A capacitor consists of two conductors which are charged.

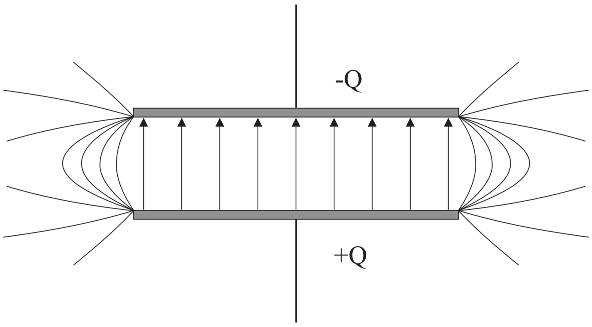

Figure 2.16: A capacitor consisting of two conducting plates which are charged with $+Q$ and $-Q$.

The potential difference between both conducting plates is:

$$V \;=\; V_+ - V_- \;=\; -\int_{-}^{+} \vec{E}(\vec{r})\; d\vec{l}$$

$\vec{E}(\vec{r})$ is proportional to Q. Therefore, the potential V is also proportional to Q. Using this proportionality the **capacitance** can be defined as follows:

$$\boxed{C \;=\; \frac{Q}{V}}$$

The units of the capacitance are Farad.
Typically a capacitor has $\mu F = 10^{-6} F$, $nF = 10^{-9} F$, or $pF = 10^{-12} F$.

$$[C] \;=\; 1\ F = 1\ \frac{C}{V} = 1\ \frac{As}{V}$$

Capacitance of a parallel plate capacitor

The surface charge is $\sigma = Q/A$, where A is the area of the capacitor plates. The corresponding electric field between the two plates is:

$$E \;=\; \frac{Q}{\varepsilon_0 A}$$

which is twice the electric field of one infinite charged conducting plate $E = \frac{\sigma}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\ \frac{Q}{A}$ (see chapter 2.3).

The electric potential of a parallel plate capacitor is therefore:

$$V = \frac{Q}{\varepsilon_0 \; A} \cdot d$$

The capacitance of a parallel plate capacitor is:

$$C = \frac{Q}{V} = \frac{A \cdot \varepsilon_0}{d}$$

The capacitance depends **only on the geometry** of the parallel plate capacitor, i.e. on the area A and the distance d between both plates.

Energy stored inside a capacitor

In order to determine the energy which is stored inside a capacitor, let us slowly charge up the capacitor with infinitesimally small additional charges dq.

$$dW = \left(\frac{q}{C}\right) \cdot dq$$

By summing up over all charges (integration from zero to Q), we obtain the energy which is stored inside a charged capacitor:

$$W = \int_0^Q \left(\frac{q}{C}\right) \, dq = \frac{1}{2} \, \frac{Q^2}{C} = \frac{1}{2} \, C \cdot V^2$$