A/ Prof. C. Ulrich

Electromagnetism PHYS2050

2 Electrostatics

2.2 Divergence and Curl of the Electric Field

The electric field, which originates from a charge density $\varrho(\vec{r}')$ is given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,\varrho(\vec{r}')\,d\vec{r}'$$

2.2.1 Divergence of the Electric Field

In the following the flux of electric field lines, which originate from a single point charge Q, is calculated:

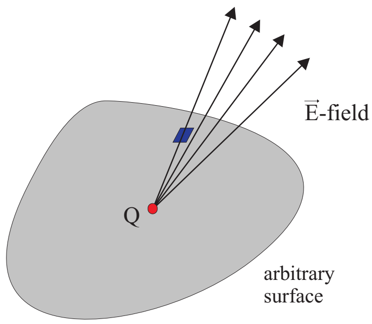


Figure 2.6: Electric field, which originate from a single point charge Q.

The flux of the electric field lines through an arbitrary surface around the point charge is:

$$\vec{E}\cdot d\vec{A} = \vec{E}\cdot\vec{n}\,dA = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\cdot\frac{\vec{r}}{|\vec{r}|}\cdot\vec{n}\,dA$$

where: $d\vec{A} = \vec{n}\,dA = \cos\Theta\,dA = r^2\,d\Omega$

$$\vec{E}\cdot\vec{n}\,dA = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\cdot\cos\Theta\,dA = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\cdot r^2\,d\Omega = \frac{1}{4\pi\varepsilon_0}Q\,d\Omega$$

$$\oint_A \vec{E}\cdot\vec{n}\,dA = \frac{1}{4\pi\varepsilon_0}\oint_A Q\,d\Omega = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} Q\sin\theta\,d\theta\,d\varphi = \frac{1}{4\pi\varepsilon_0}\cdot Q\cdot 4\pi = \frac{Q}{\epsilon_0}$$

Alternatively, consider the simplest surface around a point charge, a sphere with the area of its surface $A = 4\pi r^2$.

$$\oint_A \vec{E} \cdot \vec{n}\ dA \quad = \quad \vec{E} \cdot 4\pi r^2 \quad = \quad \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|^2} \cdot 4\pi r^2 \quad = \quad \frac{Q}{\epsilon_0}$$

In case of several source charges, or a charge density $\varrho(\vec{r})$, this equation can be written as:

$$\oint \vec{E} \cdot \vec{n}\ dA \quad = \quad \frac{Q}{\epsilon_0} \quad = \quad \frac{1}{\varepsilon_0}\int \varrho(\vec{r})\ dV$$

Using the Gauss Theorem:

$$\int \text{div}\ \vec{E}\ dV \quad = \quad \frac{1}{\varepsilon_0}\int \varrho(\vec{r})\ dV$$

The direct comparison gives:

$$\boxed{\text{div}\vec{E} \quad = \quad \frac{\varrho(\vec{r})}{\varepsilon_0}}$$

A positive charge is a **faucet** of the electric field.
A negative charge is a **drain** of the electric field.

2.2.2 Curl of the Electric Field

The curl of the electric field can be determined by using Stokes' Theorem. Therefore, we calculate the line-integral around a closed surface in an electric field. In general, a line-integral in an electric field is:

$$\int_a^b \vec{E}(\vec{r})\ d\vec{l}$$

Due to the spherical symmetry of the electric field, which originates from a point charge, it is useful to perform this calculation in spherical coordinates. The infinitesimal line element in spherical coordinates is defined as $d\vec{l}$:

$$d\vec{l} \quad = \quad dr \ + \ r\ d\theta \ + \ r\ sin\theta\ d\varphi$$

The important coordinate is the distance r from the point charge. Especially in the case of a closed integral, the integration over the two angles Θ and φ would be performed over a closed sphere.

$$\int_a^b \vec{E}(\vec{r})\; d\,\vec{l} \;=\; \frac{1}{4\pi\;\varepsilon_0} \int_a^b \frac{q}{|\vec{r}|^2}\; d\;r \;=\; -\frac{1}{4\pi\;\varepsilon_0} \left(\frac{q}{|\vec{r}|}\right)\Bigg|_a^b \;=\; \frac{1}{4\pi\;\varepsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

In case of a closed line integral one obtains:

$$\oint \vec{E}(\vec{r})\; d\vec{l} \;=\; 0$$

By using the Stokes Theorem: $\oint \vec{E}(\vec{r})\; d\vec{l} = \int \left[\nabla \times \vec{E}(\vec{r})\right]\; d\;\vec{A}$
one obtains:

$$\boxed{\nabla \times \vec{E}(\vec{r}) \;=\; 0}$$

The curl of the electric field vanishes. This holds only if there is no changing magnetic field. This equation is one of the Maxwell equations in the statical case.

Calculation of the curl of the electric field:

$$\vec{E}(\vec{R}) \;=\; \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\; \varrho(\vec{r}')\; d\vec{r}'$$

$$\text{curl}\; \vec{E}(\vec{r}) \;=\; \frac{1}{4\pi\varepsilon_0} \int \varrho(\vec{r}') \begin{pmatrix} \frac{\partial}{\partial y}\frac{z-z'}{|\vec{r}-\vec{r}'|^3} - \frac{\partial}{\partial z}\frac{y-y'}{|\vec{r}-\vec{r}'|^3} \\ \frac{\partial}{\partial z}\frac{x-x'}{|\vec{r}-\vec{r}'|^3} - \frac{\partial}{\partial x}\frac{z-z'}{|\vec{r}-\vec{r}'|^3} \\ \frac{\partial}{\partial x}\frac{y-y'}{|\vec{r}-\vec{r}'|^3} - \frac{\partial}{\partial y}\frac{x-x'}{|\vec{r}-\vec{r}'|^3} \end{pmatrix} dV$$

$$\frac{\partial}{\partial y} \frac{(z-z')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}}$$
$$= -\frac{3}{2} \frac{2(z-z')(y-y')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{(y-y')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}}$$
$$= -\frac{3}{2} \frac{2(y-y')(z-z')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{5/2}}$$

Analogously the other derivatives can be calculated. The final result proves that the curl of the electric field indeed vanishes.

$$\boxed{\text{curl}\; \vec{E}(\vec{r}) \;=\; 0}$$

2.3 The Electric Potential

curl $\vec{E}(\vec{r}) = 0$ is the condition for a conservative vector field (see definition in chapter 1.2.5). The electric field is a conservative vector field and there must be a potential whose gradient gives the electric field.

As a test, we determine the gradient of the of the function $1/|\vec{r} - \vec{r'}|$:

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{1}{|\vec{r} - \vec{r'}|} &= \frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
&= -\frac{1}{2}\cdot\frac{2(x-x')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}} \\
&= -\frac{(x-x')}{|\vec{r} - \vec{r'}|^3}
\end{aligned}$$

Similar expressions are obtained for the partial derivatives of the coordinates y and z. Therefore:

$$\text{grad}\,\frac{1}{|\vec{r} - \vec{r'}|} = -\frac{(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}$$

The electric field can be written as:

$$\begin{aligned}
\vec{E}(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3}\,\varrho(\vec{r'})\;d\,\vec{r'} \\
&= \frac{1}{4\pi\varepsilon_0}\int\limits_V -\text{grad}\frac{1}{|\vec{r} - \vec{r'}|}\,\varrho(\vec{r'})\;d\,\vec{r'} \\
&= -\text{grad}\,\frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{\varrho(\vec{r'})}{|\vec{r} - \vec{r'}|}\;d\,\vec{r'}
\end{aligned}$$

The electric field is a **conservative** vector field and the corresponding **electric potential** is:

$$\boxed{\vec{E}(\vec{r}) = -\text{grad}\;\phi(\vec{r}) \quad ; \quad \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{\varrho(\vec{r'})}{|\vec{r} - \vec{r'}|}\;d\,\vec{r'}}$$

The electric potential is a scalar function and its unit is: **Voltage**.

$$V = \phi^{(2)} - \phi^{(1)} = -\int_1^2 \vec{E}(\vec{r})\cdot d\vec{l} = \int_2^1 \vec{E}(\vec{r})\cdot d\vec{l}$$

$$[V] \;=\; [\phi] \;=\; 1\;V\;(\text{Volt}) \;=\; 1\,\frac{Nm}{As} \;=\; 1\,\frac{J}{C}$$

Lines or areas with the same potential (voltage) are equipotential lines. The electric field is always perpendicular to the equipotential lines. This is a direct consequence of the properties of the gradient.

The electric potential can be compared to the potential energy in an gravitational field.

An important consequence of the definition of the potential is, that integration in a conservative field (the electric field) is independent of the path of the integration. The integral depends only on the start and the end point.

Therefore, the line-integral over a closed curve must disappear:

$$\boxed{\oint \vec{E}(\vec{r}) \cdot d\vec{r} \;=\; 0}$$

And, as we have already pointed out, the curl of the electric field is zero:

$$\boxed{\text{curl } \vec{E}(\vec{r}) \;=\; 0}$$

The Poisson Equation

The two following equations for the electric field can be combined. The result is the Poisson equation:

$$\vec{E}(\vec{r}) \;=\; -\text{grad } \phi(\vec{r}) \qquad \text{div}\vec{E}(\vec{r}) \;=\; \frac{\varrho}{\varepsilon_0}$$

Poisson Equation of the electric potential:

$$\boxed{\text{div grad } \phi(\vec{r}) \;=\; \Delta\phi(\vec{r}) \;=\; -\frac{\varrho}{\varepsilon_0}}$$

Advantage: In case of the electric field, two equations must be solved in order to determine the electric field precisely: div $E = \varrho/\varepsilon_0$ and curl $E = 0$.
For the electric potential, only one equation, the Poisson equation, needs to be solved in order to determine the system precisely.

Example: uniformly charged spherical shell with radius R

Let us consider an uniformly charged, hollow, spherical shell with radius R. The entire charge is located ar the surface of the shell. Therefore, we can define a surface charge:

$$Q \quad = \quad 4\pi \; R^2 \; \sigma$$

Where σ is the surface charge density:

$$\sigma \quad = \quad \frac{dQ}{dA} \quad = \quad \frac{\text{charge}}{\text{area}}$$

1. Outside the shell:

At a large distance ($r \gg R$) the charged shell can be regarded as one single point charge Q, which is located at the center of the shell. The electric potential and electric field correspond to the expressions for a point charge:

$$E(r) \quad = \quad \frac{1}{4\pi\varepsilon_0} \; \frac{Q}{|r|^2} \quad ; \quad \phi \quad = \quad \frac{1}{4\pi\varepsilon_0} \; \frac{Q}{|r|}$$

2. Inside the shell:

Every surface element around a closed volume inside the shell does not surround any charge. Therefore, the electric field inside the hollow, charged shell must be zero:

$$\oint E \; dA \quad = \quad 4\pi \; r^2 \; E \quad = \quad \frac{Q}{\varepsilon_0} \quad = \quad 0 \quad \longrightarrow \quad E \quad = \quad 0$$

The potential inside a uniformly charged spherical shell is therefore constant.

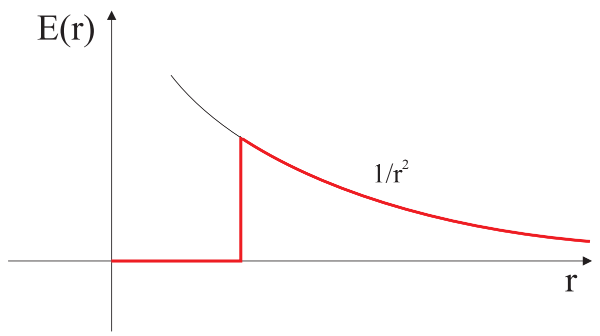


Figure 2.7: electric field of an uniformly charged spherical shell with radius R.

Example: infinite, charged plate

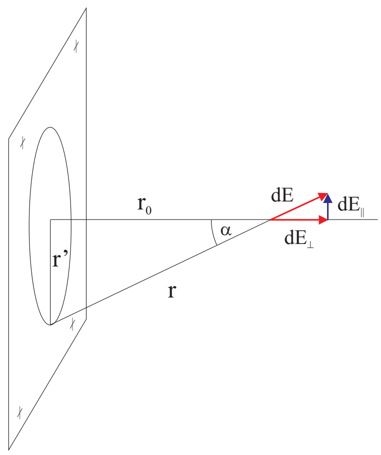


Figure 2.8: electric field of an infinite, charged plate.

In order to calculate the electric field of a infinite, charged plate, we first define the surface charge density σ:

$$dQ = \sigma \cdot dA \qquad \sigma = \frac{dQ}{dA}$$

The electric field is: $dE = \frac{1}{4\pi\varepsilon_0} \frac{dQ}{|\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|}$

The two components of the electric field, i.e. parallel and perpendicular to the charged plate, are:

$$\begin{aligned} dE_\perp &= d\vec{E} \cdot \cos\alpha = dE \cdot \cos\alpha \frac{\vec{r}}{|\vec{r}|} = dE \, \frac{\vec{r}_0}{|\vec{r}|} = \frac{1}{4\pi\varepsilon_0} \frac{\sigma dA}{|\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|} \\ dE_\parallel &= dE \cdot \sin\alpha \end{aligned}$$

Note that we have to sum (integrate) over an infinite plate. Since this is a symmetric integration (from $-\infty$ to $+\infty$) all parallel components cancel out and the resulting component of the electric field parallel to the plate must vanish: $E_\parallel = 0$.
For the component of the electric field perpendicular to the plate $E_\perp$ only the distance to the plate r_0 is important. Therefore, we can write:

$$\begin{aligned} E_\perp &= \int_A dE_\perp = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{\sigma \, dA}{|\vec{r}|^2} \frac{\vec{r}_0}{|\vec{r}|} \\ &= \int_0^{2\pi}\int_0^{\infty} \frac{\sigma}{4\pi\varepsilon_0} \frac{\vec{r}' \cdot \vec{r}_0}{|\vec{r}'^2 + \vec{r}_0^2|^{3/2}} \, d\vec{r}' \, d\varphi \\ &= \frac{\sigma \, \vec{r}_0}{2\varepsilon_0} \int_0^{\infty} \frac{\vec{r}'}{|\vec{r}'^2 + \vec{r}_0^2|^{3/2}} \, d\vec{r}' \\ &= \frac{\sigma \, \vec{r}_0}{2\varepsilon_0} \left. \frac{-1}{|\vec{r}'^2 + \vec{r}_0^2|^{1/2}} \right|_0^{\infty} = \frac{\sigma}{2\varepsilon_0} \end{aligned}$$

2.4 The Potential of an Electric Dipole

A dielectric dipole consists of two electric charges q and $-q$ which are located at a distance of a from each other. In fig. 2.9. the two charges are located at $a/2$ and $-a/2$ along the x-direction. The corresponding **dipole moment** is $\vec{p} = q \cdot \vec{r}$, i.e. $\vec{p} = (qa, 0, 0)$ and is pointing from $-q$ to q.

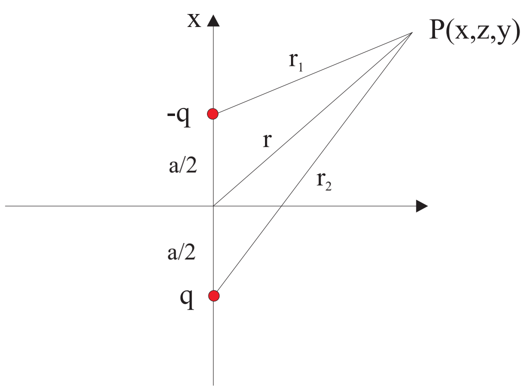


Figure 2.9: electric dipole, i.e. two charges q and $-q$ at a distance of a.

The electric potential at a certain point $P(x, y, z)$ is:

$$\phi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_2} - \frac{q}{r_1}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|r - a_x/2|} - \frac{q}{|r + a_x/2|}\right)$$

The distances $|\vec{r}|$, $|\vec{r}_1|$, and $|\vec{r}_2|$ can be calculated as follows:

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
r_1 &= \sqrt{\left(x + \frac{a}{2}\right)^2 + y^2 + z^2} = r\sqrt{1 + \frac{ax}{r^2} + \frac{a^2}{4r^2}} \\
r_2 &= \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2 + z^2} = r\sqrt{1 - \frac{ax}{r^2} + \frac{a^2}{4r^2}}
\end{aligned}$$

In case of a large distance from the electric dipole ($r \gg a$), the quadratic term under the square-root can be neglected an the square-root can be expanded into a Taylor-series.

Taylor expansion:

$$f(x) = \sum_{\nu=0}^{n} \frac{f^{\nu}(x_0)}{\nu!}(x - x_0)^{\nu} + R_n ,$$

The Taylor expansion is the development of a function into a polynomial expression. Here, $f^{\nu}(x_0)$ is the ν-th derivative of the function at the point x_0 and R_n is the Lagrangian Rest-term.

For the following approximation, we expand the square-root of the expressions for the distances into a Taylor-series and neglect quadratic and higher terms:

$$
\begin{aligned}
r_1 &\sim r\left(1+\frac{1}{2}\frac{ax}{r^2}\right)\\
r_2 &\sim r\left(1-\frac{1}{2}\frac{ax}{r^2}\right)
\end{aligned}
$$

Using this approximation the electric potential of an electric dipole can be written as:

$$
\begin{aligned}
\phi(x,y,z) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{e}{|\vec{r}-\frac{1}{2}ax|}-\frac{e}{|\vec{r}+\frac{1}{2}ax|}\right)\\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\left(\frac{1}{1-\frac{1}{2}\frac{ax}{r^2}}-\frac{1}{1+\frac{1}{2}\frac{ax}{r^2}}\right)\\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\,\frac{\frac{ax}{r^2}}{1-\frac{1}{4}(\frac{ax}{r^2})^2}
\end{aligned}
$$

In case of a large distance from the electric dipole ($r \gg a$) the quadratic term $(\frac{ax}{r^2})^2$ can be neglected. In this approximation, the potential of an electric dipole is:

$$
\phi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{qax}{|\vec{r}|^3}
$$

By replacing the electric dipole-moment $\vec{p} = (qa, 0, 0)$ the electric potential is finally:

$$
\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{|\vec{r}|^3}
$$

Using the potential of the dipole, the electric field can be calculated in cartesian coordinates a follows:

$$
\begin{aligned}
\vec{E} &= -\,\mathrm{grad}\;\phi\\
&= -\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot|\vec{r}|^3-3|\vec{r}|^2\cdot\vec{e}_r\cdot\vec{p}\cdot\vec{r}}{|\vec{r}|^6}\\
&= \frac{1}{4\pi\varepsilon_0}\frac{3\,\vec{e}_r\cdot\vec{p}\cdot\vec{e}_r-\vec{p}}{|\vec{r}|^3}
\end{aligned}
$$

Due to its radial symmetry a useful way to calculate the electric field of a dipole is using spherical coordinates. The electric field and the electric potential are independent, i.e. constant, with respect to the angle φ within the yz-plane.

Electric potential of a dipole in spherical coordinates:

$$\phi(r,\theta,\varphi) = \frac{1}{4\pi\varepsilon_0}\,\frac{\vec{p}\cdot\vec{e}_r}{|\vec{r}|^2} = \frac{1}{4\pi\varepsilon_0}\,\frac{p\cos\theta}{|\vec{r}|^2}$$

The electric field of the dipole can be determined as follows:

$$\begin{aligned}
\vec{E} &= -\text{grad}\;\phi = -\frac{\partial\phi}{\partial r}\,\vec{e}_r - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\vec{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\,\vec{e}_\varphi \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{2p\cos\theta}{|\vec{r}|^3}\,\vec{e}_r + \frac{p\sin\theta}{|\vec{r}|^3}\,\vec{e}_\theta\right) \\
&= \frac{1}{4\pi\varepsilon_0}\,\frac{3\vec{e}_r\cdot\vec{p}\cdot\vec{e}_r - \vec{p}}{|\vec{r}|^3}
\end{aligned}$$

Note that the potential of the electric dipole is proportional to $1/|r|^2$, whereas the potential of a point charge (electric monopole) is proportional to $1/|r|$.

$$\begin{aligned}
\text{electric monopole}: \quad \phi &= \frac{1}{4\pi\varepsilon_0}\,\frac{q}{|\vec{r}|} \\
\text{electric dipole}: \quad \phi &= \frac{1}{4\pi\varepsilon_0}\,\frac{p}{|\vec{r}|^2}
\end{aligned}$$

The electric field and the electric potential of a dipole are both radial symmetric around the x-direction. The electric field and the electric potential of a dipole are shown in fig. 2.10(b). Typically one refers to the field of a 'pure' electric dipole (see fig. 2.10(a)). This is allowed in case of large distances from the dipole, i.e. $|r| \gg a$ (see the approximation made above).

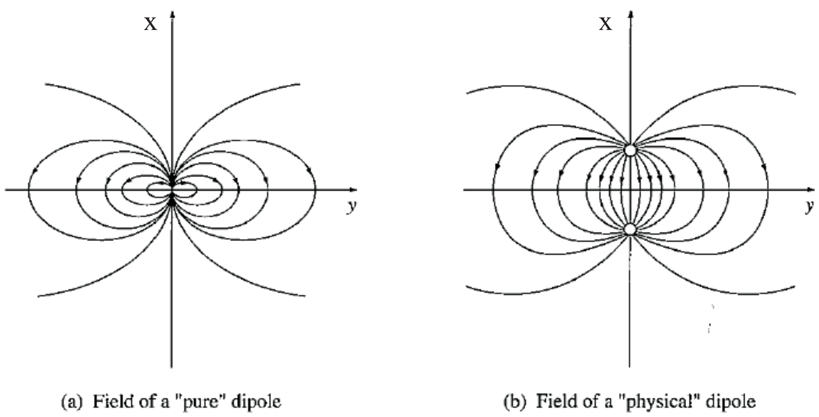


Figure 2.10: Electric field and electric potential of a dipole (from 'Introduction to Electrodynamics', David J. Griffiths, Pearson, San Francisco, 2008).

Multipole Expansion

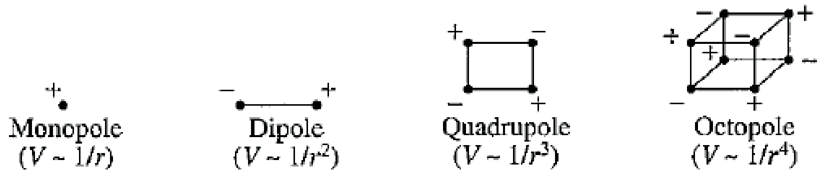
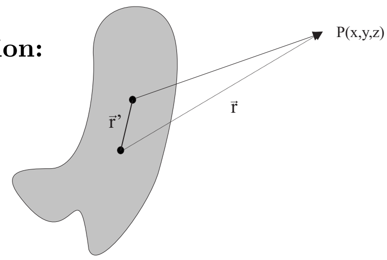


Figure 2.11: increasing proportionality $1/|\vec{r}|^n$ of the electric potential in case of an electric monopole, dipole, quadrupole, and octupole (from 'Introduction to Electrodynamics', David J. Griffiths, Pearson, San Francisco, 2008).

The most general form is a charge distribution:



$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_V \frac{\varrho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d\,\vec{r}'$$

The distance between $\vec{r}$ and $\vec{r}'$ can be calculated by using the cosinus rule:

$$|\vec{r}-\vec{r}'|^2 = \vec{r}^2 + (\vec{r})^2 - 2\vec{r}\vec{r}'\ cos\theta'$$

$$= \vec{r}^2 \left(1 + \underbrace{\left(\frac{\vec{r}'}{\vec{r}}\right)^2 - 2\left(\frac{\vec{r}'}{\vec{r}}\right)\cos\theta'}_{\epsilon=(\frac{\vec{r}'}{\vec{r}})(\frac{\vec{r}'}{\vec{r}}-2\cos\theta')} \right) = \vec{r}^2\ (1+\epsilon)$$

therefore :

$$|\vec{r}-\vec{r}'| = |\vec{r}|\ \sqrt{1+\epsilon}$$

Taylor expansion of the term $1/|\vec{r}-\vec{r}'|$:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{|\vec{r}|}\ \frac{1}{\sqrt{1+\epsilon}} = \frac{1}{|\vec{r}|}\ \left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$= \frac{1}{|\vec{r}|}\ \left(1 - \frac{1}{2}\left(\frac{\vec{r}'}{\vec{r}}\right)\left(\frac{\vec{r}'}{\vec{r}} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{\vec{r}'}{\vec{r}}\right)^2\left(\frac{\vec{r}'}{\vec{r}} - 2\cos\theta'\right)^2 \right.$$

$$\left. - \frac{5}{16}\left(\frac{\vec{r}'}{\vec{r}}\right)^3\left(\frac{\vec{r}'}{\vec{r}} - 2\cos\theta'\right)^3 + \cdots \right)$$

$$= \frac{1}{|\vec{r}|}\ \left(1 + \left(\frac{\vec{r}'}{\vec{r}}\right)\ \cos\theta' + \left(\frac{\vec{r}'}{\vec{r}}\right)^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) + \left(\frac{\vec{r}'}{\vec{r}}\right)^3\left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\right) + \cdots\right)$$

The last term is the so called **Legendre Polynomial**:

$$P_l(x) = \frac{1}{2^l\ l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \qquad \cdots \end{aligned}$$

Using the Legengre Polynomial, the electric potential of the multipole moment of an arbitrary charge distribution can be written as follows:

$$\begin{aligned} \phi(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{|\vec{r}|^{n+1}}\int_V (\vec{r}')^n\ P_n(\cos\theta')\ \varrho(\vec{r}')\ d\ \vec{r}' \\ &= \frac{1}{4\pi\varepsilon_0}\Bigg(\underbrace{\frac{1}{|\vec{r}|}\int_V \varrho(\vec{r}')\ d\ \vec{r}'}_{\text{Monopole}} + \underbrace{\frac{1}{|\vec{r}|^2}\int_V \vec{r}'\ \cos\theta'\ \varrho(\vec{r}')\ d\ \vec{r}'}_{\text{Dipole}} + \\ &\qquad \underbrace{\frac{1}{|\vec{r}|^3}\int_V (\vec{r}')^2\ \left(\frac{3}{2}\cos\theta' - \frac{1}{2}\right)\ \varrho(\vec{r}')\ d\ \vec{r}'}_{\text{Quadrupole}} + \cdots\Bigg) \end{aligned}$$

Note that the first line is the exact solution. The multipole expansion is an approximation.

Multipole Expansion:

$$\phi(\vec{r}) = \text{monopole} + \text{dipole} + \text{quadrupole} + \text{octupole} + \cdots$$