A/ Prof. C. Ulrich

Electromagnetism PHYS2050

2 Electrostatics

There are in total four different types of forces. The gravitational force describes the attractive interaction between two masses. The electromagnetic force describes the interaction between charges and magnetic interactions. The electrostatic part (Coulomb law) is especially important since this is the attractive interaction between the negatively charged electrons and the positive core of the atom.
The weak force and the strong (nuclear) force act inside the atomic core. The weak force is responsible for the β–decay of an neutron into a proton, an electron, and a neutrino. The strong nuclear force describes the attractive interaction between the particles inside the atomic core, especially between protons and neutrons.

With recent theoretical models it has became possible to combine the last three forces within the so called Grand Unified Theory (GUT).

Interaction	Strength at a distance relative to the nucleus	Length scale	Consequence
Gravitation	10^{-38}	very large	cosmos
Electromagnetic Force	10^{-2}	very large	electrons in an atom condensed matter
Weak Force	10^{-15}	very small	β-decay
Strong (Nuclear) Force	1	small	protons and neutrons inside the atomic core

2.1 The Electric Field

2.1.1 The Electric Charge

If two particles carry a charge, they experience a force, the electrostatic force (**Coulomb law**).

The unit of the charge is: Coulomb

$$[Q] \quad = \quad 1\ C = 1\ A \cdot s$$

There are both positive and negative charges.

In the case of two equal charges (++ or - -) the interaction is **repulsive**,
in the case of two unequal charges (+ - or - +) the interaction is **attractive**.

The charge is always an integer number of the elementary charge (see the experiment of R. Millikan with oil droplets in 1909). This elementary charge corresponds the charge of an electron:

$$\boxed{\mathbf{e} = 1.60217733(49) \cdot 10^{-19} C}$$

There are positive and negatively charged particles. Some examples are:

Leptons:	Electron e^-, Myon μ^-, ...
	Positron e^+, Myon μ^+, ...
Baryons:	Proton p^+, Hyperon Σ^+...
	Anti-proton p^-, Hyperon Σ^-, X^-, Ω^-, ...

Other particles carry no charge: neutron, neutrino, photon, ...

Quarks, which form the nuclear particles like the neutron and the proton, carry a charge of $+2/3$ or $-1/3$. Due to their strong binding force, Quarks cannot exist in a unbound state.

Properties of an electron:

A electron is a single point in space which has a negative charge $-1e$, a spin (1/2), and a mass:

charge	$-1.60217733(49) \cdot 10^{-19} C$
spin	1/2
mass	$m_e = 511 keV = 9.1093897 \cdot 10^{-31} kg$

This mass corresponds to:

proton	$m_p = 1836.2\ m_e$
neutron	$m_n = 1838.7\ m_e$

One has to distinguish between the three cases: a charge at rest (electrostatics), a moving charge (electrodynamics) and an accelerated charge:

Charge at rest:	electric field (Coulomb force)
Moving charge:	magnetic field (Lorentz force)
Accelerating charge:	electromagnetic wave (light wave)

The Dirac δ–Function

The electron is a single point in space (i.e. no extension). Such a single point can be described by using the δ–function:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int\limits_{-\infty}^{\infty} \delta(x)\ dx = 1$$

$$\int\limits_{-\infty}^{\infty} f(x)\delta(x - x_0)\ dx = f(x_0)$$

The Dirac δ-function is an important mathematical tool to describe single points in space. Only after integration, the δ–function gives a meaningful result, i.e. a real number. The most important property is, that the integration over a δ-function gives an area of one, i.e. $\int\limits_{-\infty}^{\infty} \delta(x)\ dx = 1$.

Approximation for the δ-function:

Especially in cases of numerical problems it is quite useful to be able to describe the δ-function by a real function which possesses approximately the same properties than the δ-function. There are different real function. One example is given below:

$$f(x) \quad = \quad \frac{2}{x}\sin(k_0 x)$$

This function converges to the Dirac δ-function if $k_0 \longrightarrow \infty$.

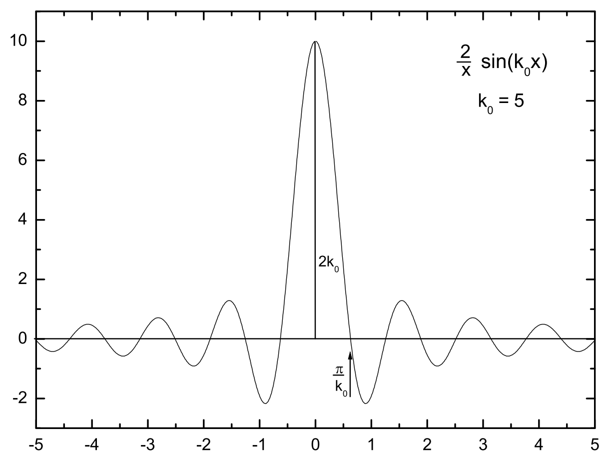

Figure 2.1.: Approximation for the Dirac δ-function.

2.1.2 The Coulomb Law

The electrostatic force acting on two charges Q_1 at $\vec{r_1}$ and Q_2 at $\vec{r_2}$ is given by the **Coulomb law**. The distance between both charges can be described by the separation vector $\vec{r} = \vec{r_2} - \vec{r_1}$.

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 \cdot Q_2}{|\vec{r_2} - \vec{r_1}|^2} \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|}$$

The Coulomb law is:

1. direct proportional to the charges Q_1 and Q_2
2. indirectly proportional to the square of the distance, i.e. $1/|\vec{r}|^2$.

Therefore, the Coulomb law has the same form as the gravitational law.

ε_0 is the dielectric constant in vacuum or electric permitivity of free space:

$$\varepsilon_0 = \frac{10^{-7}}{4\ \pi} \frac{1}{(c_0)^2} = 8.854 \cdot 10^{-12}\ \frac{A^2 s^2}{Nm^2}$$

Principle of Superposition

Let's consider a charge Q_0 at $\vec{r}$ (our probe charge) in a field of various charges with q_i, which are located ar r_i (source charges). The probe charge Q_0 experiences a Coulomb force from each individual source charge q_i.

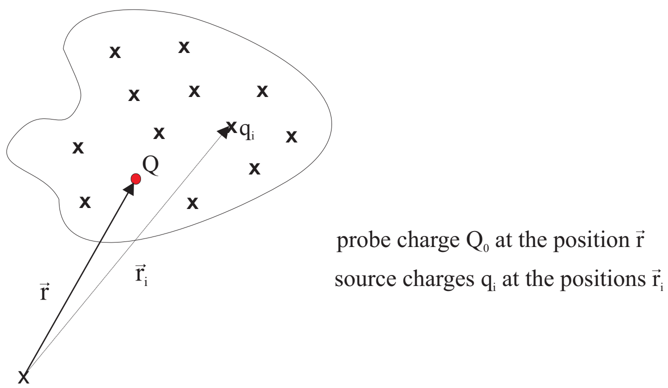

Figure 2.2: probe charge Q at $\vec{R}$ surrounded by several source charges q_i at $\vec{r_i}$.

According to the rules for the superposition of forces, the resulting force, which acts on the probe charge Q_0 can be written as:

$$\vec{F} = \sum_{i=1}^{n} \vec{F_i} = \frac{Q_0}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\vec{r} - \vec{r_i}|^2} \frac{\vec{r} - \vec{r_i}}{|\vec{r} - \vec{r_i}|}$$

2.1.3 The Electric Field

All source charges q_i together create a force on the probe charge. The final force which acts on the probe charge Q_0 is the superposition of all forces. In a more general scenario we do not need to know the distribution of all source charges in space. It is just important to know which force field they create. For this purpose we introduce the electric field $\vec{E}(\vec{r})$:

$$\boxed{\vec{F}(\vec{r}) = Q_0 \cdot \vec{E}(\vec{r})}$$

where:

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\vec{r}-\vec{r_i}|^2} \frac{\vec{r}-\vec{r_i}}{|\vec{r}-\vec{r_i}|}}$$

or in other words, the electric field is defined as:

$$\boxed{\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{Q_0}}$$

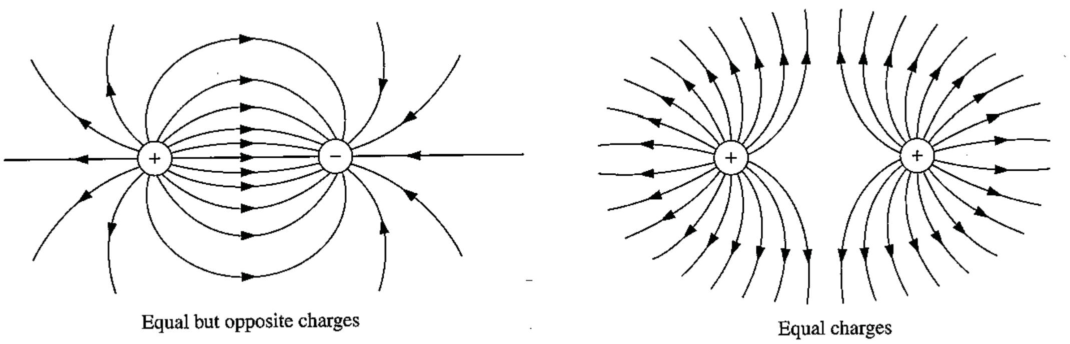

Figure 2.3: Electric field of two equal charges with opposite (left) and same (right) sign (from 'Introduction to Electrodynamics', David J. Griffiths, Pearson, San Francisco, 2008)

2.1.4 A Continuous Charge Distribution

In the following, the probe charge Q_0 experiences the Coulomb force of a continuous distribution of charges, i.e. a charge density $\varrho(\vec{r}')$.

The charge density $\varrho(\vec{r}')$ can be defined by:

$$\int \varrho(\vec{r}')\, dV' \quad = \quad q$$

where q is the total charge of the charge density $\varrho(\vec{r}')$.

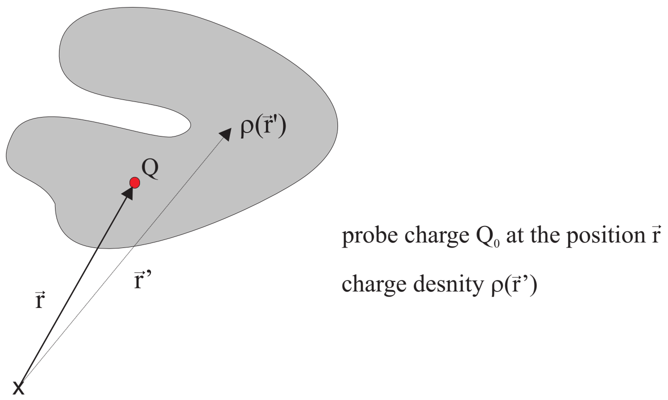

Figure 2.4: probe charge Q at $\vec{r}$ within a certain charge density $\varrho(\vec{r}')$.

The Coulomb force, which acts on the probe charge Q_0 is:

$$\vec{F}(\vec{r}) \quad = \quad \frac{Q_0}{4\pi\varepsilon_0} \int\limits_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\, \varrho(\vec{r}')\, dV'$$

Note that the integration over $dV' = d\vec{r}'$ is performed over the entire volume of the charge distribution.

Therefore, the electric field at the position $\vec{r}$, which originates from the charge density $\varrho(\vec{r}')$ is:

$$\vec{E}(\vec{r}) \quad = \quad \frac{1}{4\pi\varepsilon_0} \int\limits_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\, \varrho(\vec{r}')\, d\vec{r}'$$

In general, the electric field can be measured by moving a probe charge Q_0 through the area of the electric field and by measuring the Coulomb force, which acts on this probe charge. The resulting electric field is $\vec{E}(\vec{r}) = \vec{F}(\vec{r})/Q_0$.

2.1.5 The Electric Flux

The electric flux describes the amount of 'electric field lines' which pass through a certain surface in space.

A surface perpendicular to the electric field is:

$$d\vec{A} \quad = \quad \vec{n}\, dA$$

A general surface is defined by its tilt angle φ between the direction of the electric field and the normal $\vec{n}$ to the surface:

$$A' \quad = \quad A \cdot cos\varphi$$

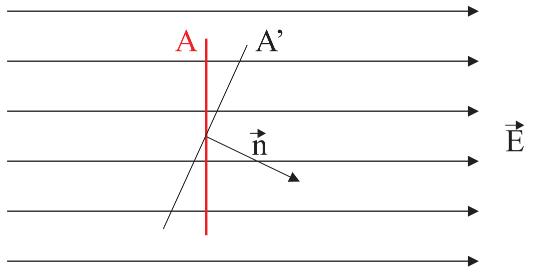

Figure 2.5: The electric flux.

The electric flux through a surface element is therefore:

$$d\Phi_{el.} \quad = \quad \vec{E} \cdot d\vec{A} \;=\; \vec{E}\,\vec{n} \cdot dA \;=\; \vec{E}\, dA \cdot cos\varphi$$

$$\boxed{\Phi_{el.} \quad = \quad \int_A \vec{E}\, dA}$$