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Electromagnetism PHYS2050

1 Vector Analysis

1.3 Integral Calculus

There are basically three different integrals in three dimensional space:

1. Line integral (path integral)
2. Surface integral (flux integral)
3. Volume integral

1.3.1 Line integral

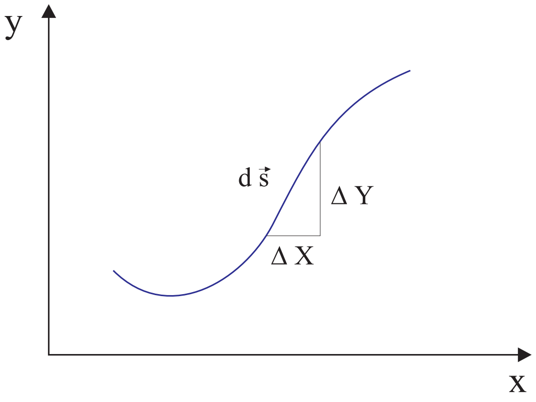


Figure 1.11: Length of a line in three dimensional space.

The length of a curve can be determined by summing over all infinitesimally small line elements ds_i:

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$\begin{aligned} L &= \sum_{i=1}^{n} \Delta s_i = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}} \cdot \Delta x_i \\ &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx \end{aligned}$$

Cartesian Coordinates

$$y = f(x) \qquad\qquad s = \int\limits_a^b \sqrt{1 + (f'(x))^2}\ dx$$

The infinitesimal displacement vector is defined as:

$$\boxed{ds \quad = \quad \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (f'(x))^2}\ dx}$$

Polar Coordinates

$$r = r(\varphi)$$
$$\varphi_0 \leq \varphi \leq \varphi_1$$

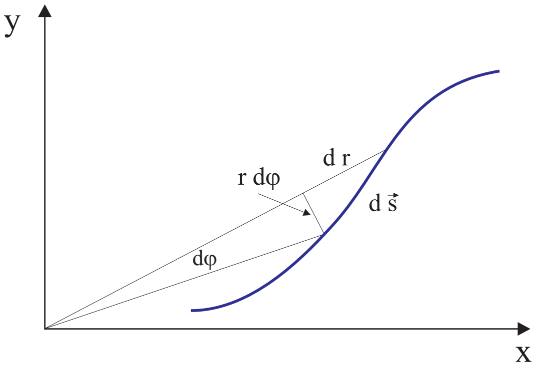


Figure 1.12: Infinitesimal displacement vector in case of polar coordinates r and φ.

$$\begin{aligned} ds &= \sqrt{(dr)^2 + (r\ d\varphi)^2} \\ s &= \int\limits_{\varphi_0}^{\varphi_1} \sqrt{(r(\varphi))^2 + (r'(\varphi))^2} d\varphi \end{aligned}$$

Example 1: circumference

The function which describes a circle can be given as follows:

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ \frac{dy}{dx} &= y'(x) = \frac{-x}{\sqrt{r^2 - x^2}} \\ 1 + (y'(x))^2 &= 1 - \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2} \end{aligned}$$

$$\begin{aligned} \mathrm{U} &= \int\limits_a^b \sqrt{1 + (f'(x))^2}\ dx = 4 \int\limits_0^r \frac{r\ dr}{\sqrt{r^2 - x^2}} = 4r \int\limits_0^1 \frac{dz}{\sqrt{1 - z^2}} \\ &= 4r \cdot \arcsin z \Big|_0^1 = 4r \cdot \frac{\pi}{2} = \mathbf{2\pi r} \end{aligned}$$

Circumference in Polar Coordinates:

$$\begin{aligned} &r(\varphi) = r \qquad \text{(const.)} \\ &0 \le \varphi < 2\pi \end{aligned}$$

$$\mathrm{U} = \int\limits_0^{2\pi} \sqrt{r^2 + r'^2}\ d\varphi = \int\limits_0^{2\pi} r\ d\varphi = \mathbf{2\pi r}$$

Example 2: The Ellipse

The equation for an ellipse is:

$$r^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\vec{x}(t) = \begin{pmatrix} a\ \cos(t) \\ b\ \sin(t) \end{pmatrix} \qquad \dot{\vec{x}}(t) = \begin{pmatrix} -a\ \sin(t) \\ b\ \cos(t) \end{pmatrix}$$

Infinitesimal displacement vector of an ellipse is:

$$\begin{aligned} ds &= \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\ dt \\ &= \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\ dt \end{aligned}$$

Circumference of an ellipse is therefore:

$$U = \int\limits_0^{2\pi} \sqrt{a^2 \sin^2(t) + b^2 \cos^2(t)}\ dt$$

This integral can only be solved numerically. This class of integrals are the so called elliptical integrals.

1.3.2 Surface Integral

An infinitesimal element of an area in three dimensional space $d\vec{A}$ is given by:

$$F = \sum_{i=1}^{3} \vec{v}(x_i, y_i, z_i)\ dA_i$$

If the surface is closed, the result can be described as the **flux** through the surface.

The direction of $d\vec{A}$ is perpendicular to the surface. Therefore, the notation $\vec{n}\ dA$ can also be used, where $\vec{n}$ is the unit vector perpendicular to the surface.

By definition, the outward direction, i.e. away from the origin, is the positive direction.

1.3.3 Gauss's theorem — divergence theorem

Let's have a closer look at a flow of a liquid (in the x-direction) through an open box.

The vector field (flow) can be described as follows::

flow: $\vec{v}(\vec{r}) = v_x$

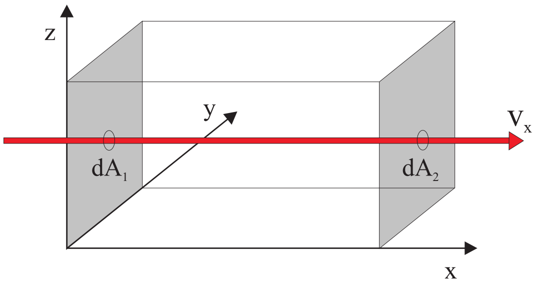


Figure 1.13: Flow of a liquid through an open box.

- The amount of liquid which enters the box through the surface dA_1 is:

  volume $v_x\ dA_1 \quad = \quad v_x \cdot dy\ dz$

- The amount of liquid which leaves the box through the surface dA_2 at $x + dx$ is:

$$v_x(x + dx, y, z)\ dy\ dz = \left[v_x + \frac{\partial v_x}{\partial x}\ dx \right]\ dy\ dz$$

- The difference is the loss or gain in the amount of liquid inside the box:

$$\frac{\partial v_x}{\partial x}\ dx\ dy\ dz$$

- We obtain the following expression for all three directions x, y, and z:

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\ dy\ dz \quad = \quad \text{div}\ \vec{v}(\vec{r})\ dV$$

- Therefore:

$$\int\limits_V \text{div}\ \vec{v}(\vec{r})\ dV \quad = \quad \int\limits_F \vec{v}(\vec{r}) \cdot \vec{n}\ dA$$

Gauss's Theorem

$$\boxed{\int\limits_V \text{div}\ \vec{v}(\vec{r})\ dV \quad = \quad \int\limits_A \vec{v}(\vec{r}) \cdot \vec{n}\ dA}$$

The change (divergence) of a vector field inside a box $\int \text{div}\ \vec{v}(\vec{r})\ dV$ corresponds to the flow through the surface of this box $\int \vec{v}(\vec{r}) \cdot \vec{n}\ dA$.

If the amount which flows into the box is equal to the amount which flows out of the box, the divergence (change = the derivative) is zero:

div $\vec{v} = 0$ no change in flow inside the box.

div $\vec{v} > 0$ More liquid is flowing out of the box, than into the box **faucet**.

div $\vec{v} < 0$ More liquid is flowing into the box, than out of box **drain**.

1.3.4 Stokes' theorem — theorem for curls

Let's assume a closed surface in space. The Stokes' theorem describes the curls (rotations) which pass through this closed surface:

$$\boxed{\oint\limits_C \vec{v}(\vec{r})\ dr \quad = \quad \iint\limits_A (\vec{n} \cdot \text{curl}\ \vec{v}(\vec{r}))\ dA}$$

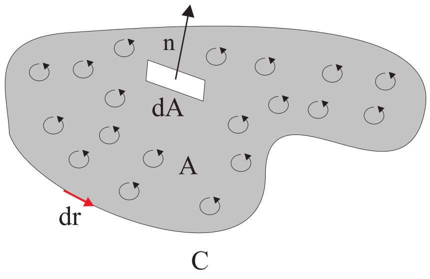


Figure 1.14: Stokes' theorem: Curls (rotations) passing through a closed surface.

The line integral around the closed surface corresponds to the area integral over all curls (flux) which pass through this surface.

The important advantage is that only the shape of the boundary (line integral) needs to be known. The shape of the surface itself does not need to be known.

The direction of the line integration is counterclockwise (right hand rule), pointing away from the origin.

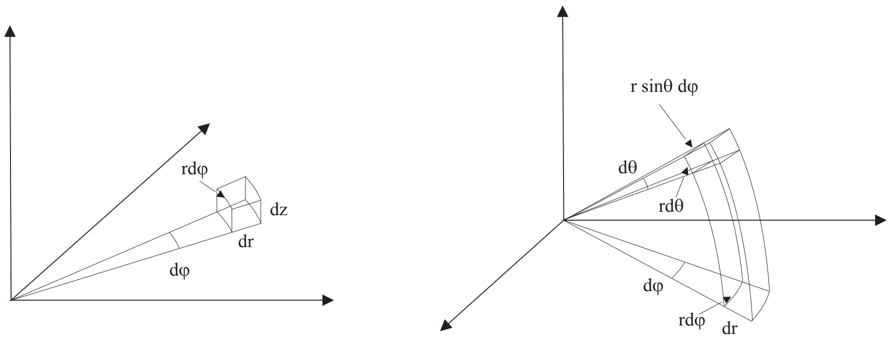
1.3.5 Volume Integral

Volume integral in cartesian coordinates

In order to describe the three dimensional integration, a volume can be divided into infinitesimally small volume elements $dv = dx\ dy\ dz$.

$$\text{volume} \quad = \quad \int_V f(x,y,z)\ dV = \iiint f(x,y,z)dx\ dy\ dy = \int_{a_1}^{a_2}\left(\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\ dz\right] dy\right) dx$$

Volume integral in spherical coordinates



cylindrical coordinates

spherical coordinates

Figure 1.15.: Volume integral in cylindrical and spherical coordinates.

The spherical coordinates r, θ, and φ are defined as follows:

$$\begin{aligned} x &= r \cdot \sin\theta \ \cdot \cos\varphi \\ y &= r \cdot \sin\theta \ \cdot \sin\varphi \\ z &= r \cdot \cos\theta \end{aligned}$$

The component for the r coordinate is dr. For the θ coordinate the component is $r \cdot d\theta$. Here, we have assumed that for small angles $\sin\theta \approx \theta$. The component for the φ coordinate is canted out of the xy-plane and therefore reduced by the amount $\sin\theta$, i.e. $r \cdot \sin\theta \; d\varphi$. In total one obtains:

$$\begin{aligned} dV &= r^2 \ \sin\theta \ dr \ d\theta \ d\varphi \\ \iiint\limits_V f(x,y,z) \ dV &= \int\limits_{R_1}^{R_2} \int\limits_{\varphi_1}^{\varphi_2} \int\limits_{\theta_1}^{\theta_2} f(r,\theta,\varphi) \ r^2 \ \sin\theta \ d\theta \ d\varphi \ dr \end{aligned}$$

Derivation of the volume element of the spherical coordinates through the determinant representation:

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{vmatrix} = r^2 \ \sin\theta$$

Summary: volume element in cartesian, cylindrical, and spherical coordinates

cartesian coordinates: $\mathbf{dV} = dx \ dy \ dz$

cylindrical coordinates: $\mathbf{dV} = r \ dr \ d\varphi \ dz$

spherical coordinates: $\mathbf{dV} = r^2 \ \sin\theta \ dr \ d\varphi \ d\theta$

Example 1: Volume of a sphere

$$
\begin{aligned}
\iiint\limits_V f(x,y,z)\ dV &= \int\limits_0^R \int\limits_0^{2\pi} \int\limits_0^{\pi} r^2 \sin\theta\ d\theta\ d\varphi\ dr = \\
&= \int\limits_0^R \int\limits_0^{2\pi} \left[-r^2 \cos\theta\right]_0^{\pi}\ d\varphi\ dr = \int\limits_0^R \int\limits_0^{2\pi} 2r^2\ d\varphi\ dr = \\
&= \int\limits_0^R 4\pi r^2\ dr = \frac{4\pi}{3} R^3
\end{aligned}
$$

Example 2: Mass of a sphere with a linearly increasing density

The density of the sphere is zero at the center an increases linearly with the radius: $\rho(r) = r/R$.

$$
\begin{aligned}
M &= \iiint\limits_V \rho(r)\ dV = \int\limits_0^R \int\limits_0^{2\pi} \int\limits_0^{\pi} \frac{r}{R}\ r^2\ \sin\theta\ d\theta\ d\varphi\ dr = \\
&= \int\limits_0^R \int\limits_0^{2\pi} \left[-\frac{r^3}{R}\ \cos\theta\right]_0^{\pi}\ d\varphi\ dr = \int\limits_0^R \int\limits_0^{2\pi} 2\frac{r^3}{R}\ d\varphi\ dr = \\
&= \int\limits_0^R 4\pi \frac{r^3}{R}\ dr = \pi R^3
\end{aligned}
$$