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Electromagnetism PHYS2050

1 Vector Analysis

1.2 Differential Calculus

1.2.1 Scalar field — Vector field

- **Scalar field**

  A scalar field $f(x, y, z)$ associates every point $P(x, y, z)$ in space with a scalar, i.e. a number: $f(\vec{x}) = f(x_1, x_2, x_3)$.

  Examples: temperature distribution (weather map), density, potential energy, altitude on a map.

  Areas or lines with $f(x, y, z) =$ conts. are equipotential areas or lines. (contour lines on a map, isothermal lines, isobar lines, etc.).
  The function $f(x, y, z)$ varies as faster, as closer the equipotential lines are.

- **Vector field**

  A vector field (function) $\vec{v}(\vec{r}) = \vec{v}(x, y, z)$ associates every point in space to a vector, i.e.

  - a direction and
  - a magnitude

  Examples: flow of water, force field, electrical field, magnetic field, wind direction on a weather card, etc.

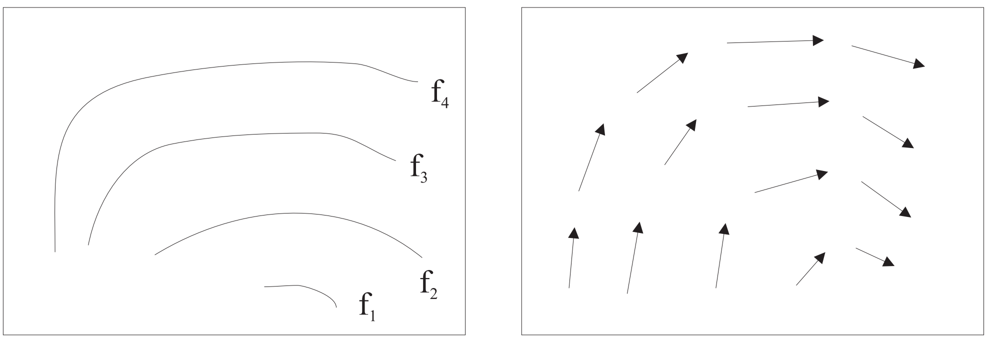

Figure 1.7: Scalar field (left) and vector field (right).

1.2.2 Ordinary Derivatives

In a three dimensional space the derivative is performed separately for each component, **partial differentiation** (partial = ∂).

$$\frac{\partial f(x,y,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} \tag{1}$$

The derivative describes the **slope** of a curve $f(x,y,z)$ at a certain point.

1.2.3 Differentiation of a Scalar Field: Gradient

How does a function $f(x,y,z)$ change, if you move by $d\vec{r} = dx + dy + dz$, i.e. what is $df(x,y,z)$ in this case?

$$\text{grad} f(x,y,z) = \nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \tag{2}$$

The gradient is the three dimensional derivative of a scalar field. The result is a vector field.

The gradient describes the change in $df(x,y,z)$ if one moved by $d\vec{r} = dx + dy + dz$:

$$df = \text{grad} f \cdot d\vec{r} = \nabla f \cdot d\vec{r} \tag{3}$$

If the direction is chosen along an equipotential line, the magnitude of $f(x,y,z)$ will not change and $df(x,y,z) = 0$. Therefore:

$$df = \nabla f \cdot d\vec{r} = 0$$

According to the rules for the scalar product of vectors, grad$f$ and $d\vec{r}$ are perpendicular to each other in this case. Therefore, the gradient of $f(x,y,z)$ is perpendicular to the equipotential lines of a scalar field. Furthermore, the gradient is pointing towards the direction of the steepest ascent. Further, the gradient is largest when the density of equipotential lines is largest.

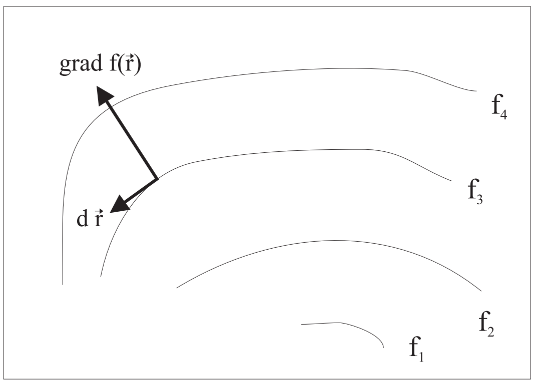

Figure 1.8: Gradient of the scalar function $f(x,y,z)$.

The Nabla–Operator

The Nabla–operator ∇ is defined as follows:

$$\nabla = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{4}$$

An operator “acts“ on a function and can also have the form of a vector.

Directional Derivative

The directional derivative is the derivative of a scalar function along a certain direction. φ is the angle between the gradient of $f(x,y,z)$ and the direction of $\vec{a}$.

$$df(x,y,z) = \text{grad} f \cdot d\vec{a} = \nabla f \cdot d\vec{a} = |\text{grad} f| \cdot |d\vec{a}| \ \cos\varphi$$

Therefore:

$$\frac{\partial f}{\partial \vec{a}}(\vec{x}_0) = \text{grad} f(\vec{x}_0) \cdot \frac{\vec{a}}{|\vec{a}|} \tag{5}$$

The result of the directional derivative is a scalar and describes the slope in the direction of the vector $\vec{a}$.

1.2.4 Differentiation of a Vector Field I: Divergence

The divergence is one of the two different derivatives of a vector field:

$$\text{div}\ \vec{v}(\vec{r}) = \nabla \cdot \vec{v}(\vec{r}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{6}$$

The result of the divergence of a vector field is a scalar field. The divergence describes how much a vector field $\vec{v}(\vec{r})$ spreads over a certain point in space.
As an example: consider an open box and a liquid is flowing in and out. If the same amount is flowing in as it is flowing out of the box, the overall amount inside the box remains the same and the change is zero: div $\vec{v} = 0$.

div $\vec{v} > 0$ more liquid is flowing out of the box: **faucet**.

div $\vec{v} < 0$ more liquid is flowing in the box: **drain**.

For example, a vector field can describe the flow of a liquid. The derivative (divergence) describes the changes at a certain point.

1.2.5 Definition of the Potential

- The derivative of a scalar field (gradient of $f(x, y, z)$) is a vector field.
- A vector field is not necessarily the gradient of a scalar field

$$\vec{v}(\vec{x}) \quad \overset{?}{\Longleftrightarrow} \quad \frac{\partial f}{\partial x}\,\vec{e}_x + \frac{\partial f}{\partial y}\,\vec{e}_y + \frac{\partial f}{\partial z}\,\vec{e}_z$$

but if this is the case:

- the vector field is **conservative**,
- and the corresponding scalar field is the **potential**.

This has several important consequences:

1. The integration in a conservative vector field is independent of the path between the start and the endpoint.
2. Therefore, an integral over a closed curve in space must give zero

$$\boxed{\oint \vec{v}(\vec{x})\, d\vec{r} \quad = \quad 0}$$

3. Furthermore:

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial v_x}\ , \qquad \frac{\partial v_x}{\partial z} = \frac{\partial v_z}{\partial v_x}\ , \qquad \frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial v_y}$$

4. As direct consequence: curl $\vec{v}(\vec{r})$ must disappear
   curl $\vec{v}(\vec{r})$ is introduced in the next chapter 1.3.6

$$\boxed{\text{curl}\ \vec{v}(\vec{x}) \quad = \quad \nabla \times \vec{v}(\vec{x}) = 0}$$

Example: Is the following vector field conservative?

$$\vec{v}(\vec{x}) \quad = \quad \begin{pmatrix} 1 & + & yz\ e^{xyz} \\ e^{y} & + & xz\ e^{xyz} \\ -\cos z & + & xy\ e^{xyz} \end{pmatrix}$$

Integration, line by line:

$$\int v_x\ dx \qquad = \textstyle\int (1 + yze^{xyz})\ dx \qquad = x + e^{xyz} + f(y, z)$$

$$\int v_y\ dy \qquad = \textstyle\int (e^{y} + xze^{xyz})\ dy \qquad = e^{y} + e^{xyz} + f(x, z)$$

$$\int v_z\ dz \quad = \textstyle\int (-\cos z + xye^{xyz})\ dz \quad = -\sin z + e^{xyz} + f(x, y)$$

The direct comparison yields:

$$f(x,y,z) \quad = \quad x + e^y - \sin z + e^{xyz}$$

The vector field is conservative and $f(x,y,z)$ is the corresponding potential.

Now let us calculate:

$$\frac{\partial v_x}{\partial y} \quad = \quad \frac{\partial v_y}{\partial v_x}$$

$$\begin{aligned}\frac{\partial v_x}{\partial y} &= z\ e^{xyz} + yz\ xz\ e^{xyz} \\ \frac{\partial v_y}{\partial x} &= z\ e^{xyz} + xz\ yz\ e^{xyz}\end{aligned}$$

These two partial derivatives are identical. The same holds for all other combinations of partial derivatives of $\vec{v}(\vec{r})$ by (x,y,z).

Therefore:

$$\begin{pmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix} \quad = \quad 0$$

This expression corresponds to the **curl** of the vector field, i.e.:

$$\text{curl } \vec{v}(\vec{x}) \quad = \quad \nabla \times \vec{v} = 0$$

1.2.6 Differentiation of a Vector Field II: Curl

The **curl** (rotation) is a differential operation on a vector field. The result is a vector field:

$$\text{curl } \vec{v} \quad = \quad \nabla \times \vec{v} = \begin{pmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix} \tag{7}$$

The **curl** describes the rotation of a vector field at a certain point.

- curl $\vec{v}$: the direction corresponds to the rotational axis,
- $|\text{curl } \vec{v}|$ the magnitude is a measure for the strength of the rotation.

Example: The flow in a river

As an example let's look at the flow in a river. At the riverbank the flow is zero, whereas the velocity of the water reaches its maximum in the center of the river.

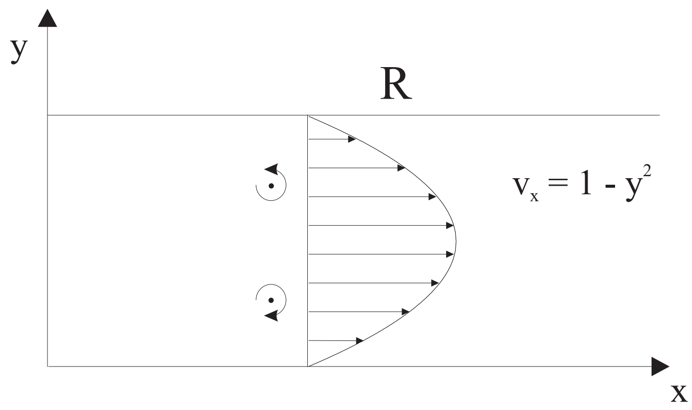

Figure 1.9: Profile of the flow in a river

The flow in a river can be described approximately by the equation of a flow through a tube (Hagen-Poiseuille):

$$v(\vec{r}) \quad = \quad \frac{\Delta P}{4l\eta}(R^2 - r^2) = \frac{g \cdot \varrho}{4\eta}(R^2 - r^2) \ ,$$

where ΔP is the difference of pressure at the length of the entire tube l, η is the viscosity and R is the diameter. For simplification, we ignore all prefactors and obtain the following profile for the flow in a river:

$$\vec{v}(r) \quad = \quad \begin{pmatrix} 1 - y^2 \\ 0 \\ 0 \end{pmatrix}$$

thus, the speed of the flow of water is $v(y_0) = (1 - y_0^2, 0, 0)$ at a distance of y_0 from the center of the river.

Divergence

$$\text{div } \vec{v} \quad = \quad \frac{\partial}{\partial x}\ (1 - y^2) + \frac{\partial}{\partial y}\ 0 + \frac{\partial}{\partial z}\ 0 \ = \ 0$$

In case of a constant flow of water (no faucet and no drain) the divergence must vanish.

Curls (rotation)

$$\text{curl } \vec{v} \quad = \quad \nabla \times \vec{v} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial y}(1 - y^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2y \end{pmatrix}$$

The resulting vector curl $\vec{v}$, i.e. the rotational axis, is pointing along the z-axis and therefore perpendicular to the xy–plane. Its magnitude is a measure for the strength of the rotation and the sign denotes the rotational sense (**right-hand-rule**):

$$
\begin{aligned}
\text{curl } \vec{v} &> 0 && \text{counterclockwise (the vector is pointing up)} \\
\text{curl } \vec{v} &< 0 && \text{clockwise (the vector is pointing town)} \\
\text{curl } \vec{v} &= 0 && \text{no rotation}
\end{aligned}
$$

Rules for the Gradient, Divergence, and Curl

In general, the rules for the gradient, divergence, and curl can be derived from the rules of the vector algebra. In this case the nabla operator ∇ is treated as a ordinary vector.

Lets assume that $f(\vec{r})$ and $g(\vec{r})$ are two scalar fields and $\vec{u}(\vec{r})$ and $\vec{v}(\vec{r})$ are two vector fields. a is a simple scalar.

The operations *grad*, *div*, and *curl* are linear

$$
\begin{aligned}
\text{grad } (f+g) &= \text{grad } f + \text{grad } g & \qquad \text{grad } a\ f &= a \text{ grad } f \\
\text{div } (\vec{u}+\vec{v}) &= \text{div } \vec{u} + \text{div } \vec{v} & \qquad \text{div } a\ \vec{f} &= a \text{ div } \vec{v} \\
\text{curl } (\vec{u}+\vec{v}) &= \text{curl } \vec{u} + \text{curl } \vec{v} & \qquad \text{curl } a\ \vec{f} &= a \text{ curl } \vec{v}
\end{aligned}
$$

or in the notation of the nabla operator:

$$
\begin{aligned}
\text{grad } (a \cdot f + b \cdot g) &= \nabla(a \cdot f + b \cdot g) = a\nabla f + b\nabla g \\
\text{div } (a \cdot \vec{u} + b \cdot \vec{v}) &= \nabla(a \cdot \vec{u} + b \cdot \vec{v}) = a\nabla\vec{u} + b\nabla\vec{v} \\
\text{curl } (a \cdot \vec{u} + b \cdot \vec{v}) &= \nabla \times (a \cdot \vec{u} + b \cdot \vec{v}) = a(\nabla \times \vec{u}) + b(\nabla \times \vec{v})
\end{aligned}
$$

Product rules:

Gradient:

$$
\begin{aligned}
\text{grad } (f\ g) &= f \cdot \text{grad } g + g \cdot \text{grad } f = f \cdot \nabla g + g \cdot \nabla f \\
\text{grad}(\vec{u} \cdot \vec{v}) &= \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u}) + (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u}
\end{aligned}
$$

Divergence:

$$
\begin{aligned}
\text{div } (f\ \vec{v}) &= f \text{ div } \vec{v} + (\text{grad } f) \cdot \vec{v} \\
\text{div } (\vec{u} \times \vec{v}) &= \vec{v} \text{ curl } \vec{u} - \vec{u} \text{ curl } \vec{v} = \vec{v}\ (\nabla \times \vec{u}) - \vec{u}\ (\nabla \times \vec{v})
\end{aligned}
$$

Curl:

$$
\begin{aligned}
\text{curl } (f\ \vec{v}) &= f \text{ curl } \vec{v} + (\text{grad } f) \times \vec{v} \\
\text{curl } (\vec{u} \times \vec{v}) &= (\vec{v} \cdot \nabla)\ \vec{u} - (\vec{u} \cdot \nabla)\ \vec{v} + \vec{u} \text{ div } \vec{v} - \vec{v} \text{ div } \vec{u}
\end{aligned}
$$

Second Derivatives:

$$
\begin{aligned}
\text{curl (grad } f) &= \nabla \times (\nabla f) = 0 \\
\text{div (curl } f) &= \nabla \cdot (\nabla \times f) = 0 \\
\text{div (grad } f) &= \nabla \nabla f = \Delta f \\
\text{curl (curl } \vec{v}) &= \text{grad(div}\vec{v}) - \Delta \vec{v}
\end{aligned}
$$

The second derivative (div(grad f) $= \Delta f$) is the **Laplace–Operator**.

$$\Delta f = \nabla \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Prove of some of the important rules:

- div (curl f) $= 0$

  $$\text{div (curl } f) = \nabla \cdot (\nabla \times f) = 0$$

  From vector algebra:

  $$\vec{a}\ (\vec{a} \times \vec{b}) = 0 \quad \text{as} \quad \vec{a} \perp (\vec{a} \times \vec{b})$$

  Divergence: The change in a vector field is pointing towards the direction of the field.
  Curl (Rotation): the rotational axis is perpendicular to the plane of the rotation.
  The scalar product of two vectors, which are perpendicular to each other, is zero.

- curl (grad f) $= 0$

  $$\text{curl (grad } f) = \nabla \times (\nabla f) = 0$$

  From vector algebra:

  $$\vec{a} \times \lambda \vec{a} = 0\ , \quad \text{because} \quad \vec{a} \parallel \lambda \vec{a}$$

  In case of two parallel vectors, the cross product is zero.

- curl (curl $\vec{v}$) = grad(div $\vec{v}$) $- \Delta \vec{v}$

  $$\text{curl (curl } \vec{v}) = \text{grad(div } \vec{v}) - \Delta \vec{v}$$

  From vector algebra:

  $$\vec{a} \times \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{c})\ \vec{b} - (\vec{a} \cdot \vec{b})\ \vec{c}$$

  therefore:

  $$\text{curl (curl } \vec{v}) = \nabla \times \nabla \times \vec{v} = \nabla(\nabla\ \vec{v}) - (\nabla \nabla)\ \vec{v}$$

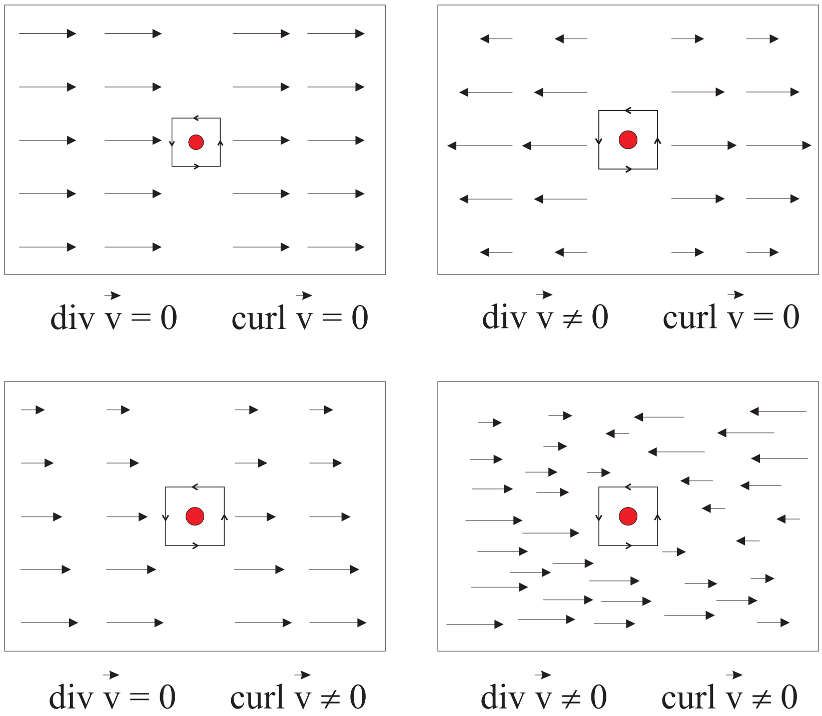

Figure 1.10: Divergence and Curl of different vector fields.

1. div $\vec{v} = 0$, curl $\vec{v} = 0$

   The first example show a constant and homogeneous flow. At every point point in this vector field the flow is constant, i.e. no change in the amount (div $\vec{v} = 0$). Furthermore, no rotation is observed (curl $\vec{v} = 0$).

2. div $\vec{v} \neq 0$, curl $\vec{v} = 0$

   The second example shows a faucet. The divergence at this point is larger than zero (div $\vec{v} > 0$). On the other side, no rotation can be observed at this point (curl $\vec{v} = 0$).

3. div $\vec{v} = 0$, curl $\vec{v} \neq 0$

   In the third example the flux is constant. Therefore, the divergence is zero (div $\vec{v} = 0$). On the other side, at every point an observer experiences a certain rotation. The curl is not zero (curl $\vec{v} \neq 0$).

4. div $\vec{v} \neq 0$, curl $\vec{v} \neq 0$

   The fourth example is a general case. It is clear that there are more long arrows than short arrows. Therefore, there must be a faucet (div $\vec{v} > 0$). Furthermore, an observer at a certain point experiences a rotation counterclockwise. Therefore, the curl must be positive: (curl $\vec{v} > 0$).

Gradient, Divergence, and Curl in Spherical Coordinates

$$\begin{aligned} x &= r\ \sin\theta\ \cos\varphi \\ y &= r\ \sin\theta\ \sin\varphi \\ z &= r\ \cos\theta \end{aligned}$$

$$r^2 = x^2 + y^2 + z^2\ ; \qquad \cot\theta = \frac{z}{\sqrt{x^2+y^2}}\ ; \qquad \tan\varphi = \frac{y}{x}$$

The unit vectors in spherical coordinates are:

$$\vec{e}_r = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \qquad \vec{e}_\theta = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \qquad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

The gradient of a scalar field can be determined as follows:

$$\boxed{\text{grad}\ f(r,\varphi,\theta) = \left(\frac{\partial f}{\partial r} \ \middle|\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi} \ \middle|\ \frac{1}{r}\frac{\partial f}{\partial\theta} \right)}$$

<u>**Derivation**</u>

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \frac{r\ \sin\theta\ \cos\varphi}{r} = \sin\theta\ \cos\varphi$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r} = \frac{r\ \sin\theta\ \sin\varphi}{r} = \sin\theta\ \sin\varphi$$

$$\frac{\partial r}{\partial z} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r} = \frac{r\ \cos\theta}{r} = \cos\theta$$

From the comparison of the resulting coordinates one realizes that the three partial derivatives of r by ∂x, ∂y, and ∂z correspond to the unit vector $\vec{e}_r$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial x}$$

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= \frac{\partial\arctan\frac{y}{x}}{\partial x} = \frac{1}{(+(\frac{y}{x})^2}(-\frac{y}{x^2}) = \frac{-y}{x^2+y^2} \\ &= \frac{-r\ \sin\theta\ \sin\varphi}{r^2\ \sin^2\theta\ \cos^2\varphi + r^2\ \sin^2\theta\ \sin^2\varphi} = \frac{1}{r\sin\theta}\ (-\sin\varphi) \end{aligned}$$

$$\begin{aligned} \frac{\partial\varphi}{\partial y} &= \frac{\partial\arctan\frac{y}{x}}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}(\frac{1}{x}) = \frac{x}{x^2+y^2} \\ &= \frac{r\ \sin\theta\ \cos\varphi}{r^2\ \sin^2\theta\ \sin^2\varphi + r^2\ \sin^2\theta\ \cos^2\varphi} = \frac{1}{r\sin\theta}\ (\cos\varphi) \end{aligned}$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\arctan\frac{y}{x}}{\partial z} = 0$$

From the comparison of the resulting coordinates one realizes that the three partial derivatives of φ by ∂x, ∂y, and ∂z correspond to the unit vector $\vec{e}_\varphi$ with an additional prefactor of $\frac{1}{r \sin\theta}$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \operatorname{arccot} \frac{z}{\sqrt{x^2+y^2}}}{\partial x} = \frac{1}{1+\frac{z^2}{x^2+y^2}} \cdot \frac{zx}{(x^2+y^2)^{3/2}} = \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$= \frac{x}{r^2} \frac{z}{\sqrt{x^2+y^2}} = \frac{x}{r^2} \cot\theta = \frac{r\sin\theta\cos\varphi}{r^2} \frac{\cos\theta}{\sin\theta} = \frac{1}{r} (\cos\varphi \;\cos\theta)$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \operatorname{arccot} \frac{z}{\sqrt{x^2+y^2}}}{\partial y} = \frac{1}{1+\frac{z^2}{x^2+y^2}} \cdot \frac{zy}{(x^2+y^2)^{3/2}} = \frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$= \frac{y}{r^2} \frac{z}{\sqrt{x^2+y^2}} = \frac{y}{r^2} \cot\theta = \frac{r\sin\theta\sin\varphi}{r^2} \frac{\cos\theta}{\sin\theta} = \frac{1}{r} (\sin\varphi \;\cos\theta)$$

$$\frac{\partial \theta}{\partial z} = \frac{\partial \operatorname{arccot} \frac{z}{\sqrt{x^2+y^2}}}{\partial z} = \frac{-1}{1+\frac{z^2}{x^2+y^2}} \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{-\sqrt{x^2+y^2}}{r^2} = \frac{-r\sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi}}{r^2} = \frac{1}{r} (-\sin\theta)$$

From the comparison of the resulting coordinates one realizes that the three partial derivatives of θ by ∂x, ∂y, and ∂z correspond to the unit vector $\vec{e}_\theta$ with an additional prefactor of $\frac{1}{r}$.

This is the proof for the expression of the gradient in spherical coordinates. The expressions for the divergence, curl, and the Laplace operator can be derived in a similar way. The results are given below:

$$\operatorname{grad} f(r,\varphi,\theta) = \left(\frac{\partial f}{\partial r} \;\middle|\; \frac{1}{r\,\sin\theta} \frac{\partial f}{\partial \varphi} \;\middle|\; \frac{1}{r} \frac{\partial f}{\partial \theta} \right)$$

$$\operatorname{div} \vec{v}(\vec{r}) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta} \frac{\partial v_\varphi}{\partial \varphi}$$

$$\Delta\, f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \tan\theta} \frac{\partial f}{\partial \theta}$$

$$\operatorname{curl} \vec{v}(r,\varphi,\theta) = \nabla \times \vec{v}(r,\varphi,\theta) = \begin{pmatrix} \frac{1}{r\sin\theta} \left(\frac{\partial}{\partial \theta}(v_\varphi \sin\theta) - \frac{\partial v_\theta}{\partial \varphi} \right) \\ \frac{1}{r\sin\theta} \frac{\partial v_r}{\partial \varphi} - \frac{1}{r} \frac{\partial}{\partial r}(r v_\varphi) \\ \frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \end{pmatrix}$$