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Electromagnetism PHYS2050

1 Vector Analysis

1.1 Vector Algebra

Several physical properties are defined by one single number, i.e. a scalar:

temperature	T	K
density	ϱ	kg/m^3
output	P	Watt = J/s = $kg \cdot m^2/s^3$

Other physical properties possess besides their magnitude also a direction. They can be described in the form of a vector:

force	$\vec{F}$	Newton; $N = kg \cdot m/s^2$
velocity	$\vec{v}$	m/s
angular momentum	$\vec{M} = \vec{r} \times \vec{F}$	Newtonmeter = N $\cdot$ m

1.1.1 Definition of a Vector

- A **scalar** is a number: c
- A **vector** $\vec{a}$ is defined through its:
  - length (i.e. magnitude of the vector) $|\vec{a}|$
  - direction $\vec{e}_{\vec{a}} = \vec{a}/|\vec{a}|$

vector $\quad \vec{a} = |\vec{a}| \cdot \vec{e}_{\vec{a}}$

Note that a vector has a magnitude an a direction, but not a location. It can be moved to any starting point in space.

some special vectors:

- Unit vector: its magnitude is **1**.
- Zero vector: $\vec{0} = (0,0,0)$

A vector can be described within a certain coordinate system:

$$\vec{a} = (a_1, a_2, a_3) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{1}$$

$$= a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \sum_i a_i \cdot \vec{e}_i$$

in case of a rectangular coordinate system (cartesian coordinates), the unit-vectors of the coordinate system are defined as follows: $\vec{e}_1 = (1,0,0)$; $\vec{e}_2 = (0,1,0)$; and $\vec{e}_3 = (0,0,1)$.

1.1.2 Multiplication with a Scalar

If $\vec{a}$ is a vector and $c \in R$ a scalar, the product $c\,\vec{a}$ is a vector with is stretched or squeezed by the factor $|c|$. If $c \leq 0$ the direction of the vector is inverted.

$$c \cdot \vec{a} = c \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{pmatrix} \tag{2}$$

$$(c + d) \cdot \vec{a} = c \cdot \vec{a} + d \cdot \vec{a} \tag{3}$$

$$c \cdot (\vec{a} + \vec{b}) = c \cdot \vec{a} + c \cdot \vec{b} \tag{4}$$

1.1.3 Addition and Subtraction of Vectors

The addition or subtraction of vectors is performed for every coordinate. The resulting vector can also be determined through the sin or cos laws using the angle between both vectors.

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \tag{5}$$

$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \tag{6}$$

The subtraction of vectors can be understood as the addition of the inverted vector $-\vec{b}$.

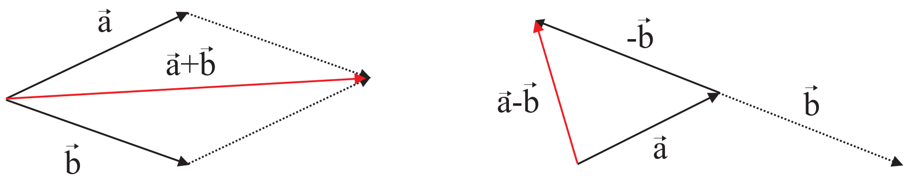

Figure 1.1: Addition and subtraction of vectors.

1.1.4 Magnitude of a vector

The magnitude of a vector is its length:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^3} \tag{7}$$

Note:

$$|\vec{a}| = |-\vec{a}| \tag{8}$$

$$|c\,\vec{a}| = |c| \cdot |\vec{a}| \tag{9}$$

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| \tag{10}$$

$$|\vec{a} - \vec{b}| = |\vec{b} - \vec{a}| \tag{11}$$

1.1.5 Linear-combination of vectors

A particle is in equilibrium when all forces acting on this particle cancel out each other:

$$\sum_i \vec{a}_i = \vec{0} \tag{12}$$

i.e. this particle is at rest or moves with a constant velocity in one direction.

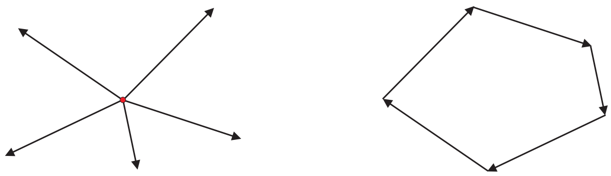

Figure 1.2: A particle in its equilibrium position and the corresponding addition of all vectors.

Linear-combination of Vectors

Linear-combination

$$\vec{x} \quad = \quad c_1 \cdot \vec{a}_1 + \cdots + c_n \cdot \vec{a}_n = \sum_{i=1}^{n} c_i \cdot \vec{a}_i \tag{13}$$

The vectors $\vec{a}_1 \cdots \vec{a}_n$ are **linearly independent**, if the zero-vector can only be constructed if $c_i = 0$. Linearly independent vectors define the space and its dimensionality, i.e. they are the basis of the space.

If two vectors are linearly dependent, they are pointing towards the same direction.

If three vectors are linearly dependent, they are lying within the same plane.

More than three vectors in a three dimensional space are always linearly dependent.

1.1.6 Scalar Product of Vectors

The scalar product of the two vectors $\vec{a}$ and $\vec{b}$ is a scalar:

$$\vec{a} \cdot \vec{b} \quad = \quad a_1 b_1 \; + \; a_2 b_2 \; + \; a_3 b_3 \tag{14}$$

$$= \quad |\vec{a}| \; |\vec{b}| \; \cos \varphi \tag{15}$$

where φ is the angle between both vectors.

Note

$$\vec{a} \cdot \vec{b} \quad = \quad 0$$

Either one of the vectors is the zero vector or both vectors are perpendicular to each other: $\vec{a} \perp \vec{b}$, i.e. $\not\sphericalangle \, (\vec{a}, \vec{b}) = 90^0$.

$$\vec{a} \cdot \vec{b} \quad = \quad |\vec{a}| \cdot |\vec{b}|$$

Both vectors are collinear: $\vec{a} || \vec{b}$, i.e. $\not\sphericalangle \, (\vec{a}, \vec{b}) = 0^0$.

Product Rules

$$
\begin{aligned}
\text{commutative law} \quad & \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} && (16)\\
& (r\vec{a})\cdot\vec{b} = r(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(r\vec{b}) && (17)\\
\text{distributive law} \quad & \vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} && (18)\\
& \vec{a}\cdot\vec{a} = \vec{a}^2 = |\vec{a}|^2 \geq 0 && (19)\\
\text{Note!} \quad & (\vec{a}\cdot\vec{b})\cdot\vec{c} \neq \vec{a}\cdot(\vec{b}\cdot\vec{c}) && (20)
\end{aligned}
$$

Important Properties

$$
\text{angle between } \vec{a} \text{ and } \vec{b} \qquad \cos\varphi = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}|\cdot|\vec{b}|} \tag{21}
$$

$$
|\vec{a}\cdot\vec{b}| \leq |\vec{a}|\cdot|\vec{b}| \tag{22}
$$

$$
|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}| \tag{23}
$$

Orthogonal, Orthonormal

- Two vectors which are perpendicular to each other are **orthogonal**.
- If they are unit vectors in addition (i.e. $\vec{n}_0 = \vec{n}/|\vec{n}|$), then they are called **orthonormal**.

$$
\begin{array}{lllll}
\vec{e}_i\cdot\vec{e}_j = 0 & \text{if} & i \neq j & \text{i.e.} & \vec{e}_i \perp \vec{e}_j & \text{orthogonal}\\
\vec{e}_i\cdot\vec{e}_j = 1 & \text{if} & i = j & \text{1.e.} & |\vec{e}_i| = |\vec{e}_j| = 1 & \text{normalized}
\end{array}
$$

Kronecker delta function

$$
\vec{e}_i\cdot\vec{e}_j = \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \tag{24}
$$

Example: derivation of the formula for the magnitude of a vector:

$$
\begin{aligned}
|\vec{a}|^2 &= \vec{a}\cdot\vec{a} = (\sum_i a_i\vec{e}_i)\cdot(\sum_i a_i\vec{e}_i) = \sum_i\sum_j (a_i\cdot\vec{e}_i)\cdot(a_j\cdot\vec{e}_j) \qquad (25)\\
&= \sum_i\sum_j a_i a_j\cdot\vec{e}_i\vec{e}_j = \sum_i\sum_j a_i a_j\cdot\delta_{i,j} = \sum_i a_i^2 = a_1^2+a_2^2+a_3^2
\end{aligned}
$$

1.1.7 The Cross Product

Definition of the Cross Product

The result of the cross product of two vectors $\vec{a}$ and $\vec{b}$ is a vector:

$$\vec{c} \quad = \quad \vec{a} \times \vec{b} = \vec{e}_{\vec{c}} \cdot |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi \tag{26}$$

- φ is the angle between bothy vectors.
- The resulting vector is perpendicular to the plane of the two vectors $\vec{a}$ and $\vec{b}$.
- For $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ the **right hand rule** applies.
- If the two vectors $\vec{a}$ and $\vec{b}$ collinear,
  the cross product gives the zero vector.

$$\vec{a} \times \vec{b} = \vec{0} \quad \Longrightarrow \quad \vec{a}||\vec{b}$$

- The magnitude of the resulting vector $\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \sin\varphi$ corresponds to the area of the $\vec{a}$ und $\vec{b}$ parallelogram between them.

$$\vec{a} \times \vec{b} \quad = \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{27}$$

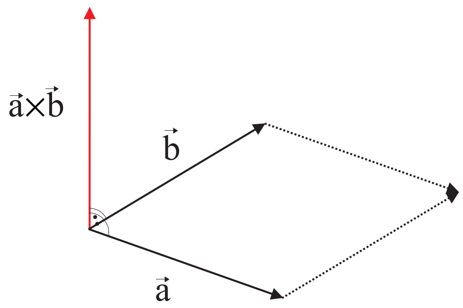

Figure 1.3: Cross product of two vectors.

Important properties:

$$\vec{a} \times \vec{b} \quad = \quad -(\vec{b} \times \vec{a}) \tag{28}$$

$$r\ (\vec{a} \times \vec{b}) \quad = \quad (r\ \vec{a}) \times \vec{b} = \vec{a} \times (r\ \vec{b}) \tag{29}$$

$$\vec{a} \times (\vec{b} + \vec{c}) \quad = \quad \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \tag{30}$$

double cross product $\quad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \qquad (31)$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

Note! $\quad \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0} \qquad (32)$

Lagrange – identity $\quad (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \qquad (33)$

$(\vec{a} \times \vec{b})^2 = \vec{a}^2 \vec{b}^2 - (a \cdot b)^2 \qquad (34)$

Note! $\quad (\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

The product $(\vec{a} \cdot \vec{b}) \times \vec{c}$ is invalid since $(\vec{a} \cdot \vec{b})$ is a scalar and $\vec{c}$ is a vector.

The triple product:

$$\begin{aligned} V &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= (\vec{b} \times \vec{c}) \cdot \vec{a} \\ &= (\vec{c} \times \vec{a}) \cdot \vec{b} \quad = \quad \langle \vec{a}, \vec{b}, \vec{c} \rangle \end{aligned} \tag{35}$$

gives the volume of the of the parallelepiped which is described by the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

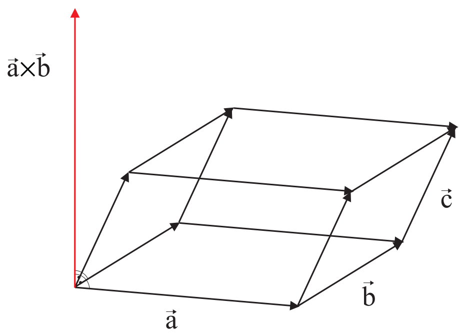

Figure 1.4: triple product: Volume of the parallelepiped.

Cross product of two cross products

$$\begin{aligned} (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= \langle \vec{a}, \vec{c}, \vec{d} \rangle \cdot \vec{b} - \langle \vec{b}, \vec{c}, \vec{d} \rangle \cdot \vec{a} = \langle \vec{a}, \vec{b}, \vec{d} \rangle \cdot \vec{c} - \langle \vec{a}, \vec{b}, \vec{c} \rangle \cdot \vec{d} \\ (\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) &= \langle \vec{a}, \vec{b}, \vec{c} \rangle \cdot \vec{b} \end{aligned} \tag{36}$$

Calculation of the cross product by using determinants

The cross product can also be calculated by using the notation of a determinant. In this case, the first column are the unit vectors of the corresponding coordination system.

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \\ &= \vec{e}_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - \vec{e}_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + \vec{e}_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= \vec{e}_1(a_2 b_3 - a_3 b_2) + \vec{e}_2(a_3 b_1 - a_1 b_3) + \vec{e}_3(a_1 b_2 - a_2 b_1) \\ &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \end{aligned} \tag{37}$$

Summary

Scalar Product

$$\vec{a} \cdot \vec{b} \quad = \quad |\vec{a}| \cdot |\vec{b}| \cdot cos\varphi \tag{38}$$

Cross Product

$$\vec{a} \times \vec{b} \quad = \quad \vec{e}_{\perp(\vec{a},\vec{b})} \cdot |\vec{a}| \cdot |\vec{b}| \cdot sin\varphi \tag{39}$$

Important properties:

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| \tag{40}$$

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| \cdot |\vec{b}| \tag{41}$$

$$V = \langle \vec{a}, \vec{b}, \vec{c} \rangle = (\vec{a} \times \vec{b}) \cdot \vec{c} \tag{42}$$

	Scalar Product	Cross Product
commutative law	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
associative law	$r(\vec{a} \cdot \vec{b}) =$ $= (r\vec{a}) \cdot \vec{b} = \vec{a} \cdot (r\vec{b})$	$r \cdot (\vec{a} \times \vec{b}) =$ $= (r \cdot \vec{a}) \times \vec{b} = \vec{a} \times (r \cdot \vec{b})$
distributive law	$\vec{a} \cdot (\vec{b} + \vec{c}) =$ $= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$	$\vec{a} \times (\vec{b} + \vec{c}) =$ $= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
Note!	$(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$	$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

1.2 Coordinate Systems

1.2.1 Cartesian Coordinates

Cartesian coordinates are defined by using orthonormal vectors:

$$\begin{array}{llllll} \vec{e}_i \cdot \vec{e}_j = 0 & \text{if} & i \neq j & \text{i.e.} & \vec{e}_i \perp \vec{e}_j & \text{orthogonal} \\ \vec{e}_i \cdot \vec{e}_j = 1 & \text{if} & i = j & \text{i.e.} & |\vec{e}_i| = |\vec{e}_j| = 1 & \text{normalized} \end{array}$$

In the three dimensional space the unit vectors which span the space are:

$$\vec{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{43}$$

A coordinate system does not need to be cartesian. Any noncollinear vectors can serve as a basis of the space. A different choice of the coordinate system can be an advantage, depending on the particular problem. In such a case one coordinate system can be transformed into another.

1.2.2 Cylindrical Coordinates

For certain problems, like rotations around an axis, its appropriate to describe the three dimensional space in cylindrical coordinates. In this case every position in space can be described as lying on the surface of a cylinder. The three coordinates which describe the position in space are:

- Distance $r \geq 0$, i.e. the distance from the rotational axis.
- Angle φ, is the rotational angle around the axis. The rotational direction is counterclockwise ($0 \leq \varphi < 2\pi$).
- Height z corresponds to the height of the cylinder ($-\infty < z < \infty$).

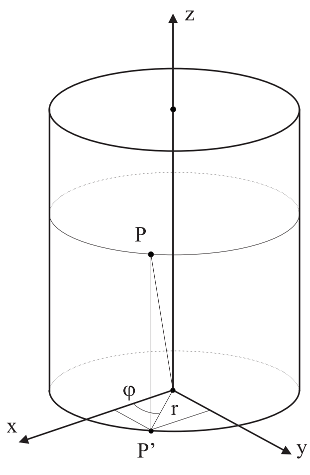

Figure 1.5.: Cylindrical coordinates r, φ, and z.

Transformation into cartesian coordinates

$$x = r \cdot \cos\varphi \quad (44)$$
$$y = r \cdot \sin\varphi \quad (45)$$
$$z = z \quad (46)$$

$$r = \sqrt{x^2 + y^2} \quad (47)$$
$$\cos\varphi = x/\sqrt{x^2 + y^2} \quad (48)$$
$$\sin\varphi = y/\sqrt{x^2 + y^2}$$
$$z = z \quad (49)$$

1.2.3 Spherical Coordinates

Three dimensional space can be also described using spherical coordinates. The most prominent example is the position on earth given in longitude and latitude
(Sydney, Opera House: Latitude: -33.8567 S, Longitude: 151.2153 E).

The spherical coordinates are:

- Distance $r \geq 0$ from the origin O, $r \geq 0$.
- Angle φ, rotation within the xy-plane, i.e. around the z-axis, $0 \leq \varphi < 2\pi$.
- Angle θ, i.e. the canting angle out of the xy-plane,
  $-\pi/2 \leq \theta \leq \pi/2$ or $0 \leq \theta \leq \pi$,
  depending on the various definition of θ.

The values (r, φ, θ) are the **spherical coordinates** of the point P $(\vec{r}, \vec{\varphi}, \vec{\theta})$.
At the origin $\vec{O}$, only r is defined as $r = 0$ and φ, θ are arbitrary.

Transformation into cartesian coordinates

From fig. 1.6 one can derive the following expressions:

$$x = |\overline{OP'}| \cdot \cos\varphi$$
$$y = |\overline{OP'}| \cdot \sin\varphi$$
$$|\overline{OP'}| = r \cdot \cos\theta$$

from these equations one can derive the following transformation rules:

$$x = r \cdot \cos\theta \cdot \cos\varphi \quad (50)$$
$$y = r \cdot \cos\theta \cdot \sin\varphi \quad (51)$$
$$z = r \cdot \sin\theta \quad (52)$$

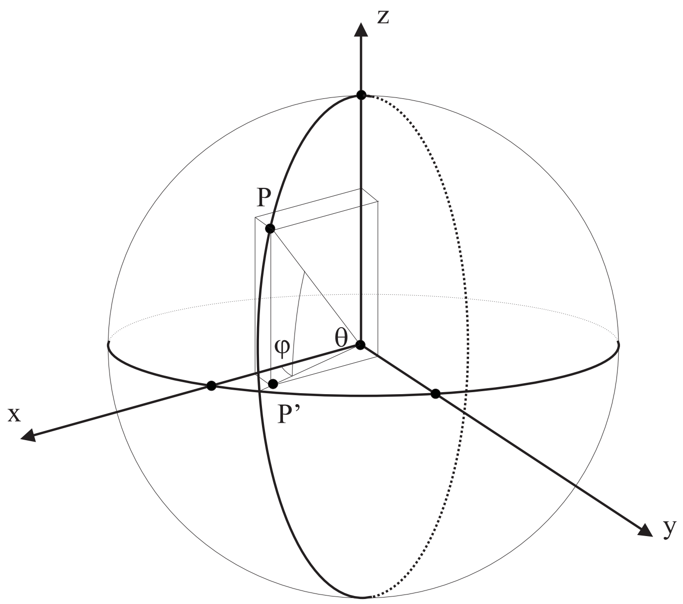

Figure 1.6.: Spherical coordinates r, φ and θ of the point P.

$$
\begin{aligned}
x^2 + y^2 + z^2 &= r^2 \\
x/\sqrt{x^2+y^2} &= \cos\varphi & z/\sqrt{x^2+y^2} &= \sin\theta/\cos\theta = \tan\theta \\
y/\sqrt{x^2+y^2} &= \sin\varphi & y/x &= \sin\varphi/\cos\varphi = \tan\varphi
\end{aligned}
$$

Furthermore, the spherical coordinates can be determined from the cartesian coordinates as follows:

$$r = x^2 + y^2 + z^2 \tag{53}$$

$$\tan\theta = z/\sqrt{x^2+y^2} \tag{54}$$

$$\tan\varphi = y/x \tag{55}$$