PHYS2939 Electromagnetism
(Electrical Engineering)

Part 2:
Magnetic Fields and Materials
Maxwell’s Equations and Waves.
Griffiths Chapters 5, 6, 7, sect. 9.1, 2.

A/Prof. Michael Box (room 102B, OMB)

Lecture 9
Maxwell’s Displacement Current
Maxwell’s Equations in Matter
Lecture 10
Boundary Conditions
Electromagnetic Waves
Lecture 11
Plane Waves
Energy; Dielectrics

Electromagnetism before Maxwell

So far we have obtained 4 fundamental equations which govern electric and magnetic fields. Apart from Faraday's law, they were developed for statics.

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad (\textit{Gauss' law})$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (\textit{no name})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (\textit{Faraday's law})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad (\textit{Ampere's law})$$

That was the state of knowledge when Maxwell arrived on the scene. These equations are <u>not</u> consistent. If we take the divergence of Faraday's law

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0$$

using the second equation. This is as it should be: the divergence of a curl is always zero.

But now if we take the divergence of Ampere's law

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} = -\mu_0 \, \partial \rho / \partial t \qquad (6.16)$$

which in general (*non static*) cases, is <u>not</u> zero.

Maxwell's Displacement Current

The right side of (6.16) is not zero when it should be. Using Gauss' law, we may manipulate this as follows:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\varepsilon_0 \nabla \cdot \mathbf{E}\right) = -\nabla \cdot \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

This means that if we add a term $\varepsilon_0 \partial \mathbf{E} / \partial t$ to **J** in Ampere's law, then its divergence will be zero:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad (6.17)$$

This is now the fourth of the governing equations for electromagnetism, which are known as Maxwell's equations.

A physical example illustrates the problem. Consider current flowing to the plate of a capacitor. On one side of the plate a current is flowing – on the other there is none. If we draw an Amperian loop at the point of the capacitor plate, what is the enclosed current?

The extra term we have added makes no difference to magnetostatics, of course, and will only be significant in situations where **E** is changing <u>very</u> rapidly. Normally this term is swamped by the current, **J**.

Maxwell called the extra term the displacement current

$$\mathbf{J}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad (6.18)$$

although it is really not a current at all.

We may easily convert (6.17) to integral form, namely

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} + \mu_0 I_{d,enc} \qquad (6.19)$$

where $$I_{d,enc} = \varepsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{a} \qquad (6.20)$$

(This allows us to do problems like Example 6 before.)

The displacement current term adds symmetry to the set of equations governing **E** and **B**, which now read

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

We see that, not only does a changing magnetic field create an electric field (Faraday’s law), but a changing electric field also creates a magnetic field.

Why did we find one so easily, but not the other? The simple reason is that producing currents to create a magnetic field is easy, as no charge separation is required. Electric fields are actually harder to make (and of course ε_0 is a very small number).

Look again at the sources of the fields. A changing **E** creates a **B**, and vice versa. Also, electric fields are produced by charges, and magnetic fields by currents - flows of charge. But we don't see magnetic fields being produced by magnetic ‘charges’, and electric fields produced by flows of magnetic charges.

This is the first time in history that a physical concept/ law/fact was *predicted* using mathematics, rather than *observed* experimentally: it marks a turning point in the history of science. Many (experimental) physicists of Maxwell’s generation were not impressed!

To complete the theory of *classical electrodynamics* we need to add one further equation, the Lorentz force:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Note that the continuity equation for electric charge (i.e. conservation of charge) is included in Maxwell’s equations – take the divergence of (6.17) and use Gauss’ law.

In material media the situation is just so complicated (accounting for every electron), that these equations are of little <u>practical</u> use. Nevertheless, they are still the fundamental description of electromagnetic phenomena.

Maxwell's Equations in Matter

Earlier we studied static electric and magnetic fields in matter, and introduced some additional vector fields. Now we are doing dynamics, we must reexamine, and modify, these results.

The electric polarization field, **P**, results in the accumulation of bound charges:

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad and \qquad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

Any time variation in **P** will involve a flow of these bound charges, $\mathbf{J}_p$ say, which must now be added to **J**. Consider a small piece of material, with surface bound charges at each end (see Fig. 7.45, page 329).
If **P** increases, so does σ_b, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$

and so

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \tag{6.21}$$

Note that $\mathbf{J}_p$ is in no way related to the bound current, $\mathbf{J}_b$, which is an entirely magnetic phenomenon, involving electron orbits and spin. $\mathbf{J}_p$ is the result of linear motion of the electric polarization charges. If **P** points to the right, and increases, each positive charge moves a little to the right, and each negative charge moves a little to the left. While this is happening, there is, in effect, a current flowing.

Note that $\mathbf{J}_p$ and ρ_b obey a continuity equation:

$$\nabla \cdot \mathbf{J_p} = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{P}\right) = -\frac{\partial \rho_b}{\partial t}$$

Thus $\mathbf{J}_p$ is essential to conserve bound charge.

We may now split the total charge density in two

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

and the current density in three

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \qquad (6.22)$$

Gauss' law may now be written

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right)$$

or $\quad \nabla \cdot \mathbf{D} = \rho_f$

where $\quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad (6.23)$

just as in the static case (i.e. no additional terms).

Turn now to Ampere's law – as corrected by Maxwell – and insert (6.22):

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

or $\quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \qquad (6.24)$

where $\quad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \qquad (6.25)$

Faraday's law, and $\nabla \times \mathbf{B} = 0$ do not involve charges or currents, so they will be unaffected. Thus we arrive at the following set of equations:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{6.26 a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.26 b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.26 c}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{6.26 d}$$

These equations are "convenient", as they only involve free currents and charges. However, they involve four vector fields – **E**, **D**, **B** and **H** – with no adequate connections. Equations (6.23) and (6.25) only serve the purpose of introducing two new fields, **P** and **M**.

In the case of linear media, we have the *phenomenological* equations

$$\mathbf{P} = \varepsilon_0\, \chi_e\, \mathbf{E} \quad \Rightarrow \quad \mathbf{D} = \varepsilon\, \mathbf{E}$$

and

$$\mathbf{M} = \chi_m\, \mathbf{H} \quad \Rightarrow \quad \mathbf{H} = \mu^{-1}\, \mathbf{B}$$

but the values of χ_e and χ_m (or ε, μ) are experimental (phenomenological), and not obtained from classical Physics. The relation $\mathbf{J} = \sigma\, \mathbf{E}$ is also phenomenological.

Note that **D** is called the electric 'displacement', which explains Maxwell's name "displacement current":

$$\mathbf{J}_d = \partial \mathbf{D} / \partial t$$

Boundary Conditions

When we work with material media, we need to be aware that the fields **E**, **B**, **D** and **H** are discontinuous across boundaries, especially if there is a surface charge density, σ, and/or surface current density, **K**.

Equations (6.26) can tell us about these, if we convert them to integral form

$$\left.\begin{aligned} i)\quad & \oint_s \mathbf{D}\cdot d\mathbf{a} = Q_{f(enc)} \\ ii)\quad & \oint_s \mathbf{B}\cdot d\mathbf{a} = 0 \end{aligned}\right\} \quad \textit{integrated over any closed surface, } S$$

$$\left.\begin{aligned} iii)\quad & \oint_L \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{a} \\ iv)\quad & \oint_{\mathbf{L}} \mathbf{H}\cdot d\mathbf{l} = I_{f(enc)} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{a} \end{aligned}\right\} \quad \textit{for any surface, } S\textit{, bounded by a closed loop } L$$

We may now use the standard approaches – a wafer-thin Gaussian pill-box straddling the surface, or a very thin Amperian loop straddling the surface – we may obtain the following results:

$$D_{1\perp} - D_{2\perp} = \sigma_f \qquad (6.27)$$

$$B_{1\perp} - B_{2\perp} = 0 \qquad (6.28)$$

$$\mathbf{E}_{1\Pi} - \mathbf{E}_{2\Pi} = 0 \qquad (6.29)$$

$$\mathbf{H}_{1\Pi} - \mathbf{H}_{2\Pi} = \mathbf{K}_f \times \hat{\mathbf{n}} \qquad (6.30)$$

In the case of linear media, we may express these results just in terms of **E** and **B**:

(6.27) $\Rightarrow$ $$\varepsilon_1 E_{1\perp} - \varepsilon_2 E_{2\perp} = \sigma_f$$

(6.30) $\Rightarrow$ $$\mu_1^{-1}\mathbf{B}_{1\prod} - \mu_2^{-1}\mathbf{B}_{2\prod} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

(6.31)

Potentials in Electrodynamics

Since **B** remains divergenceless, we may still write

$$\mathbf{B} = \nabla \times \mathbf{A}$$

But the curl of **E** is no longer zero. Using the above in Faraday’s law, we now have

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{A}\right)$$

i.e. $$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

Now we do have a vector whose curl vanishes, and so it may be written as the gradient of a scalar potential:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

i.e. $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

This reduces to the old form in static situations.

Note that we still have some flexibility in choosing both V and **A**, but these choices are now ‘coupled’.