

# **PHYS2939 Electromagnetism**

## **(Electrical Engineering)**

Part 2:

Magnetic Fields and Materials

Maxwell's Equations and Waves.

Griffiths Chapters 5, 6, 7, sect. 9.2.

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Lecture 7.

Ohm's Law, EMF

Faraday's Law

Example

Lecture 8.

Inductance

Energy in Magnetic Fields

## Ohm's Law

Electric current is the flow of electric charge, and comes from forces exerted on free charges. This force comes from electric fields, and we usually have

$$\mathbf{J} = \sigma \mathbf{E} \quad (6.0)$$

where  $\sigma$  is the *conductivity* of the material.

If this is uniform, the continuity equation gives

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = -\frac{1}{\sigma} \frac{\partial \rho}{\partial t} = 0$$

since charge density cannot change with time. Hence  $\mathbf{E}$  is constant, and any excess charge is on the surface.

Consider a cylindrical wire, length  $L$ , area  $A$ . Let the potential between the ends be  $V$ . Current flowing is

$$I = J A = \sigma E A = \sigma A \frac{V}{L} \equiv \frac{V}{R}$$

Thus (6.0) implies Ohm's law.

Power is force  $\times$  velocity. Integrate over the wire:

$$P = \int \rho \mathbf{E} \cdot \mathbf{v} d\tau = \int \mathbf{E} \cdot \mathbf{J} d\tau = \sigma \int E^2 d\tau$$

For a wire of length  $L$  and area  $A$ , this becomes

$$P = \sigma E^2 L A = \sigma V^2 A / L = V^2 / R$$

This is just an alternative derivation of a well known result for circuit elements.

## Electromotive Force

Question: What drives current? Answer; batteries etc.

But outside the battery, why doesn't charge build up somewhere? If it did, this local charge density would provide an additional force on charges at the head of the 'bunch' to move faster. Thus, electrostatic forces will act to smooth out the charges.

This process will then act to transmit the EMF of the battery: current acts to transmit the driving force, and with it the current – it self-perpetuates.

Ohm's law comes from

$$\mathbf{J} = \sigma \mathbf{f}$$

where  $\mathbf{f}$  is the force per unit charge. This force has two components; the source  $\mathbf{f}_s$  (battery etc) and the electrostatic force  $\mathbf{E}$ , which smooths the flow:

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

The electromotive force,  $\varepsilon$ , is defined to be the line integral of  $\mathbf{f}$  around the circuit:

$$\varepsilon \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

since the line integral of an electrostatic field is zero.

Finally, EMF determines the amount of current flowing:

$$\varepsilon = \oint \mathbf{f} \cdot d\mathbf{l} = \frac{1}{\sigma} \oint \mathbf{J} \cdot d\mathbf{l} = \frac{1}{\sigma} \oint \frac{I}{A} d\mathbf{l} = I \frac{L}{A\sigma} = IR$$

## Motional EMF

It takes a source of EMF to maintain a current against a resistance. A battery converts chemical energy into electrical energy. A generator converts kinetic energy, or “motional energy”, into electrical energy.

Consider fig 7.10 from Griffiths. In the shaded region is a uniform magnetic field,  $\mathbf{B}$ , pointing into the page. The wire loop has resistance  $R$ . If we pull this loop to the right with speed  $v$ , charges within the segments  $ab$ ,  $bc$  and  $ad$  experience magnetic forces,  $qvB$ . The force on all segments is up the page.

Hence, only the force on segment  $ab$  is parallel to the wire, and thus capable of driving a current clockwise around the loop. The EMF is now

$$\mathcal{E} = \oint \mathbf{f}_{mag} \cdot d\mathbf{l} = v B h \quad (6.1)$$

Magnetic forces do no work: but work is being done. By what? The agent doing the pulling on the loop.

Once currents start to flow, the charges in segment  $ab$  will have some average vertical velocity  $\mathbf{u}$ , in addition to their horizontal velocity  $\mathbf{v}$ . Thus, the magnetic force they experience will have a component  $quB$  to the left, resisting the pulling of the loop. Thus an equivalent force will need to be maintained to the right, resulting in work being done.

There is a better way of expressing this EMF. Consider the magnetic flux through the loop:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \quad (6.2)$$

which in this case becomes

$$\Phi = B h s$$

As the loop moves out of the field region,  $\Phi$  decreases:

$$\frac{d\Phi}{dt} = B h \frac{ds}{dt} = -B h v$$

But this is just (6.1) with a minus sign:

$$\varepsilon = -\frac{d\Phi}{dt} \quad (6.3)$$

This version, the flux rule, for the motional EMF has the great advantage that it is applicable to any shaped loop, moving in any direction, and through uniform or non-uniform magnetic fields, including all of these things varying in time. (Griffiths has the full proof.)

We need to think through the sign question, which concerns both the direction of  $\varepsilon$ , and of  $d\mathbf{a}$ . Answer; use the right hand rule.

## Faraday's Law

Eq. (6.3) is very general.  $\Phi$  can change because we move a bar magnet near a conducting loop. It can also change if we move the loop, not the magnet. It is the *relative* motion that is important.

If the loop moves, its charges have velocity, and so experience a magnetic force, which translates to an EMF. But if the magnet moves, the charges can't experience a magnetic force – and yet an EMF *is* produced: both experiment, and (6.3), show this.

So what does exert the force on these charges? The simplest answer is an electric field. However, this can't be an electrostatic field (which can't generate EMFs). Remember that it is being produced by a non-static magnetic field. We are forced to conclude that *a changing magnetic field induces an electric field*.

In the cases we have looked at, this induced electric field produces the EMF, so we must have

$$\oint \mathbf{E} \cdot d\mathbf{l} = \varepsilon = -\frac{d\Phi}{dt} \quad (6.4)$$

This is Faraday's law in integral form. We want to convert it to differential form; use Stokes' theorem:

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{a} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

i.e.  $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (6.5)$

This result applies whenever  $\mathbf{B}$  changes in time, regardless of how or why. A electric field always accompanies a changing magnetic field. Of course, in static situations, (6.5) reduces to  $\nabla \times \mathbf{E} = 0$ .

To keep track of the sign in Faraday's law, we may resort to Lenz's law:

*If a current flows, it will be in such a direction that the magnetic field it produces tends to counteract the change in flux that produced the EMF.*

Nature is conservative – “inertial” – and prefers to maintain the status quo: or at least to try. Note that if the opposite law applied, the universe would be very unstable – small changes could be rapidly amplified. It would also produce something for nothing. (Pity)

(6.5) is mathematically the same as Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

so it can be ‘solved’ with the same mathematical form:

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{(\partial \mathbf{B} / \partial t) \times \hat{\mathbf{r}}}{r^2} d\tau = -\frac{d}{dt} \left\{ \frac{1}{4\pi} \int \frac{\hat{\mathbf{B}} \times \hat{\mathbf{r}}}{r^2} d\tau \right\} = -\frac{\partial \mathbf{A}}{\partial t} \quad (6.6)$$

The integral form of Faraday's law, is like the integral form of Ampere's law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

It may be solved if the symmetry is right.

### Example (6)

A uniform magnetic field,  $\mathbf{B}(t)$ , points straight up, filling the shaded circular region. If it is changing with time, find the induced  $\mathbf{E}$  as a function of  $r$ .

Draw an Ampere-type loop of radius  $r$ , and apply Faraday's law in integral form:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E 2\pi r = -\dot{\Phi} = -\pi r^2 \dot{B}$$

$$\therefore E = -\frac{1}{2} r \frac{dB}{dt}$$

The direction of  $\mathbf{E}$  is circumferential: if  $\mathbf{B}$  is increasing,  $\mathbf{E}$  runs clockwise.

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Question: when is magnetostatics not magnetostatics?  
We've been using magnetostatic results like Ampere to calculate  $\mathbf{B}$  fields, and then letting them be non-static. In fact, we can get away with this unless things are changing extremely rapidly – GHz frequencies!  
We are working in a 'quasistatic' regime.

## Inductance

Consider two arbitrary conducting loops of wire, at rest. If a steady current,  $I_1$ , flows around loop 1, it will produce a magnetic field  $\mathbf{B}_1$ . Some of the field lines of this field will pass through loop 2, producing a flux  $\Phi_2$ .

Since 
$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}$$

and 
$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2$$

we may write 
$$\Phi_2 = M_{21} I_1 \tag{6.7}$$

That is,  $\Phi_2$  is proportional to  $I_1$ , and the constant of proportionality (which only depends on the geometry),  $M_{21}$ , is the mutual inductance.

(This is quite similar to capacitance, and only a few simple cases can be done.)

Let's derive an important, 'neater' expression:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

$$(4.13) \Rightarrow \quad \mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{r}$$

$$\therefore \Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left( \oint \frac{d\mathbf{l}_1}{r} \right) \cdot d\mathbf{l}_2$$

$$\therefore M_{21} = \frac{\mu_0}{4\pi} \iint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \quad (6.8)$$

and because of the obvious symmetry of (6.8) we see that

$$M_{21} = M_{12} \equiv M$$

From (6.8) we can appreciate that  $M$  is a purely geometrical quantity, depending only on the shapes of the two loops, and their relative separation.

Whether or not we can evaluate (6.8) does not alter its symmetry, which tells us that

the flux through loop 1, when a certain current  $I$  flows in loop 2,

is identical to the flux in loop 2, when the same current flows in loop 1.

Suppose now that the current in loop 1 varies in time. Thus the flux in loop will 2 to vary, inducing an EMF:

$$\varepsilon_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI}{dt} \quad (6.9)$$

That is, changing the current in loop 1 induces an EMF in loop 2, and presumably a current as well.

What if we consider just a single loop?

If a current flows in it, it will produce a magnetic field, and hence a flux, through itself. Again, there must exist a linear relation between flux and current:

$$\Phi = LI \quad (6.10)$$

This constant of proportionality is called the (self-) inductance. If  $I$  changes, so will  $\Phi$ , inducing an EMF:

$$\varepsilon = -L \frac{dI}{dt} \quad (6.11)$$

Because of the minus sign (or Lenz's law), this is referred to as a "back" EMF: it is such as to oppose any change in the current in the loop.

Unit of inductance: henries (H):  $1 \text{ H} = 1 \text{ volt-sec/Amp}$ .

### Example (10)

A short solenoid, of length  $l$ , radius  $R$ , and  $n_1$  turns per unit length, lies on the axis of a very long solenoid, of  $n_2$  turns per unit length. Find  $M$ .

Because of the symmetry, we could start with either solenoid. However, we know the field of a very long solenoid from a previous example:

$$B = \mu_0 n_2 I$$

(constant); so the flux through the inner solenoid is

$$B \pi R^2 = \mu_0 n_2 I \pi R^2$$

through each loop. Since there are  $n_1 l$  total turns on this solenoid, the total flux is

$$\Phi = \mu_0 \pi R^2 n_1 n_2 \ell I$$

$$\therefore M = \mu_0 \pi R^2 n_1 n_2 \ell$$

Look at Example 11 – self-inductance of a rectangular toroidal coil – and Example 12 – the basic  $LR$  circuit.

## Energy in Magnetic Fields

If we have a conducting loop with no current flowing in it, and then try to cause a current to flow, this will be opposed – at least initially – by the back EMF. Thus it takes work, or energy, to start a current flowing. (Work is also required to keep the current flowing against the resistance, but that's a separate issue.) This energy is of fixed amount (for a given current etc), and is recoverable.

The work done per unit charge, against the back EMF, in traversing the circuit once, is  $-\varepsilon$  (– because you are working against  $\varepsilon$ ). The amount of charge flowing per unit time is  $I$ , so the work done per unit time is

$$\frac{dW}{dt} = -\varepsilon I = LI \frac{dI}{dt}$$

If we start from no current, and build up to a final current  $I$ , the total work is found by integrating:

$$W = \frac{1}{2} LI^2 \quad (6.12)$$

The time required doesn't matter, only the geometry, as reflected in  $L$ , and the final current  $I$ .

We now seek to obtain a far more general result.

Using the vector potential,  $\mathbf{A}$ ,

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{a} = \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_c \mathbf{A} \cdot d\mathbf{l}$$

where  $C$  is the perimeter of the loop, and  $S$  any surface bounded by  $C$ . Combining with (6.10) gives

$$\Phi = LI = \oint_c \mathbf{A} \cdot d\mathbf{l}$$

$$\therefore W = \frac{1}{2} I \oint_c \mathbf{A} \cdot d\mathbf{l}$$

$$\text{i.e. } W = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I}) d\ell$$

This result may be easily extended to volume currents:

$$W = \frac{1}{2} \int_{vol} (\mathbf{A} \cdot \mathbf{J}) d\tau \quad (6.13)$$

Now use Ampere's law,  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ , to remove  $\mathbf{J}$ :

$$W = \frac{1}{2\mu_0} \int_v \mathbf{A} \cdot (\nabla \times \mathbf{B}) d\tau$$

We will integrate this by parts, making use of the rule

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\therefore \mathbf{A} \cdot (\nabla \times \mathbf{B}) = B^2 - \nabla \cdot (\mathbf{A} \times \mathbf{B})$$

$$\begin{aligned}
\therefore W &= \frac{1}{2\mu_0} \left\{ \int B^2 d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B}) d\tau \right\} \\
&= \frac{1}{2\mu_0} \left\{ \int_{vol} B^2 d\tau - \oint_{surf} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \right\} \quad (6.14)
\end{aligned}$$

Now what volume are we integrating over? It must include all of  $\mathbf{J}$ , but including more will not change (6.13). Hence it cannot change (6.14). If we make this volume large enough,  $\mathbf{A}$  and  $\mathbf{B}$  will decrease at such a (combined) rate that the surface integral will be zero. So eventually we may ignore this term, leaving

$$W = \frac{1}{2\mu_0} \int_{all\ space} B^2 d\tau \quad (6.15)$$

This result shows clear parallels with the electrical case, so we often say that energy is stored in the magnetic field, with a density of  $(2\mu_0)^{-1} B^2$ . It certainly took energy to create the field, and ultimately it makes little difference whether we regard this energy as stored in  $\mathbf{B}$  or stored in  $\mathbf{J}$  (or stored in  $\mathbf{E}$  or in  $\rho$ ).

Compare the two sets of formulae for stored energy:

$$\begin{aligned}
W_e &= \frac{1}{2} \int \rho V d\tau = \frac{1}{2} \epsilon_0 \int E^2 d\tau \\
W_m &= \frac{1}{2} \int (\mathbf{A} \cdot \mathbf{J}) d\tau = \frac{1}{2\mu_0} \int B^2 d\tau
\end{aligned}$$

## Example

A coaxial cable carries a current  $I$  on its inner surface of radius  $a$ , and  $-I$  on its outer surface of radius  $b$ . Find the energy stored in a section of length  $d$ .

Ampere's law (in integral form) shows that  $\mathbf{B}$  is zero everywhere except between the cylinders, where

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

So energy density  $= \frac{1}{2\mu_0} B^2 = \frac{\mu_0 I^2}{8\pi^2 r^2}$

$$\therefore W = d \int_a^b \left( \frac{\mu_0 I^2}{8\pi^2} \right) \frac{1}{r^2} 2\pi r dr = d \frac{\mu_0 I^2}{4\pi} \ln(b/a)$$

Note that since  $W = 0.5 LI^2$ , we may actually use this approach to find  $L$  in tricky circumstances.