

PHYS2939 Electromagnetism

(Electrical Engineering)

Part 2:

Magnetic Fields and Materials

Maxwell's Equations and Waves.

Griffiths Chapters 5, 6, 7, sect. 9.2.

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Lecture 5.

Magnetostatic Fields in Matter

Bound Currents and Magnetization

The Auxiliary Field H

Lecture 6.

Linear Media

Ferromagnetism

Magnetization

If you put a piece of stuff in an electric field it tends to become polarized (conductors an exception).

If you put a piece of stuff in a magnetic field it tends to become magnetically polarized – or magnetized.

This time, we have three types of material:

paramagnets: magnetization is parallel to \mathbf{B} ;

diamagnets: magnetization is antiparallel to \mathbf{B} ;

ferromagnets: magnetization remains after \mathbf{B} removed.

Paramagnets are attracted into a non-uniform field, diamagnets are repelled.

Follow the pattern of the electrostatic case:

Define the **magnetization**, \mathbf{M} , as

$\mathbf{M} \equiv$ magnetic dipole moment per unit volume

For now we don't care how or why, only that it exists.

For para- and diamagnets, \mathbf{M} can be very weak; for ferromagnets, \mathbf{M} can be very large.

Magnetization is a result of electron orbital motion.

Normally these orbits, and the corresponding magnetic dipoles, are randomly oriented, and the material is unmagnetized. But in the presence of an external magnetic field, the torque it exerts tends to align these dipoles, giving the material an overall magnetization.

This is the origin of *paramagnetism*.

What about *diamagnetism*?

Consider an electron orbit, which constitutes a current:

$$I = -\frac{e}{T} = -\frac{ev}{2\pi r}$$

where T is the orbital period, and v the electron speed. The corresponding dipole moment is

$$m = \pi r^2 I = -\frac{1}{2} evr$$

Electrons orbit because $\mathbf{F}_{\text{Coulomb}} = \mathbf{F}_{\text{centripetal}}$,

i.e.
$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} = m_e \frac{v^2}{r}$$

However, a mag field provides an additional force:

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} + ev'B = m_e \frac{v'^2}{r} \quad v' > v$$

$$\therefore ev'B = \frac{m_e}{r} (v'^2 - v^2) = \frac{m_e}{r} (v' + v)(v' - v)$$

$$\therefore \Delta v = v' - v \approx \frac{erB}{2m_e}$$

This leads to a change in magnetic dipole moment of

$$\Delta \mathbf{m} = -\frac{1}{2} e(\Delta v) r \mathbf{k} = -\frac{e^2 r^2}{4m_e} \mathbf{B}$$

Thus the change in \mathbf{m} is opposite to \mathbf{B} , so the substance acquires a magnetization opposite to that imposed – this is diamagnetism.

Bound Currents

The vector potential of a single magnetic dipole is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \quad (4.17')$$

Now our object has a dipole moment $\mathbf{M}d\tau$ in each $d\tau$.

$$\therefore \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M} \times \hat{\mathbf{r}}}{r^2} d\tau \quad (5.1)$$

We manipulate this as follows:

$$\nabla \frac{1}{r} = \frac{\hat{\mathbf{r}}}{r^2}$$

(taken with respect to source coordinates within $d\tau$)

$$\therefore \mathbf{A} = \frac{\mu_0}{4\pi} \int \left(\mathbf{M} \times \nabla \frac{1}{r} \right) d\tau$$

We may use vector calculus results to rearrange this to

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{r} (\nabla \times \mathbf{M}) d\tau - \int \nabla \times \left(\frac{1}{r} \mathbf{M} \right) d\tau \right\} \\ &= \frac{\mu_0}{4\pi} \int \frac{1}{r} (\nabla \times \mathbf{M}) d\tau + \frac{\mu_0}{4\pi} \oint \frac{1}{r} (\mathbf{M} \times d\mathbf{a}) \end{aligned} \quad (5.2)$$

The first term looks like potential of a volume current,

$$\mathbf{J}_b = \nabla \times \mathbf{M} \quad (5.3)$$

Second term looks like potential of a surface current,

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \quad (5.4)$$

where $\hat{\mathbf{n}}$ is a unit normal vector.

Thus we may write for the vector potential

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{vol} r^{-1} \mathbf{J}_b d\tau + \frac{\mu_0}{4\pi} \oint_{surf} r^{-1} \mathbf{K}_b da \quad (5.5)$$

Now, instead of evaluating the potential due to all the individual dipoles, we can evaluate the potential due to these two bound currents – just as for the volume and surface charges in the electrostatic case.

Physical Interpretation. Read the section in Griffiths.

Example:

Consider a cylinder of length L and radius R , carrying a uniform magnetization \mathbf{M} parallel to its axis.

Since \mathbf{M} is constant, the bound volume current is zero;

but $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \hat{\mathbf{z}} \times \hat{\mathbf{r}} = M \hat{\phi}$

This is just the same ‘surface current’ as a solenoid.

If L is much larger than R , we may now ‘solve’ this problem using the standard long solenoid techniques:

check that $\mathbf{B} = \mu_0 K_b \hat{\mathbf{z}} = \mu_0 M \hat{\mathbf{z}}$

If L is not large compared to R , then we can divide it into a series of slices, take the result for the field along the axis of a current loop (Example 3 from Lecture 1), and integrate. (The result is more complex.)

The Auxiliary Field, \mathbf{H}

Now consider a fully realistic situation, which might contain both free currents, \mathbf{J}_f , and bound currents, \mathbf{J}_b :

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

This separation is really just a convenience. Ampere's law concerns the total current, \mathbf{J} , so we may write

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

i.e.
$$\nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

Just as with the electric displacement current, \mathbf{D} , it is convenient to introduce a new auxiliary vector field – the “magnetic intensity”:

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad (5.6)$$

Now, in terms of \mathbf{H} , Ampere's law becomes

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad (5.7)$$

or
$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f,enc} \quad (5.7')$$

$$\left\{ \text{c.f.} \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{and} \quad \nabla \cdot \mathbf{D} = \rho_f \quad \right\}$$

Clearly \mathbf{H} is the magnetic equivalent of \mathbf{D} .

However, in practice it is readily determined from the free currents, whereas in the electric case we measure potential difference (not free charges), and hence \mathbf{E} .

Linear Media

For paramagnetic and diamagnetic materials, the magnetization, \mathbf{M} , is directly proportional to the applied field, \mathbf{B} . In a similar, but not quite parallel analogy to linear dielectric media^{*}, we choose to write this proportionality as

$$\mathbf{M} = \chi_m \mathbf{H} \quad (5.8)$$

(Remember that \mathbf{H} is easier to measure than \mathbf{B} .)

χ_m is called the magnetic susceptibility, and is a dimensionless quantity. It is positive for paramagnetic materials, and negative for diamagnetic materials.

Combining with (5.6) we have

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + \chi_m) \mathbf{H} \quad (5.9)$$

i.e.
$$\mathbf{B} = \mu \mathbf{H} \quad (5.10)$$

where
$$\mu = \mu_0 (1 + \chi_m) \quad (5.11)$$

is called the permeability of the material.

For a linear medium, the volume bound current density

is
$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f \quad (5.12)$$

So unless free current flows through the material, all the bound current will be on the surface.

$$* \left\{ \text{c.f. } \mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \mathbf{D} = \epsilon \mathbf{E} \quad \epsilon = \epsilon_0 (1 + \chi_e) \right\}$$

Example

An infinite solenoid with n turns per unit length, and carrying a current I , is filled with linear material of susceptibility χ_m . Find the magnetic field inside.

Start by determining \mathbf{H} from (5.7') – just as for an empty solenoid we immediately get:

$$\mathbf{H} = n I \hat{\mathbf{k}}$$

Then from (5.9) we get

$$\mathbf{B} = \mu_0 (1 + \chi_m) n I \hat{\mathbf{k}}$$

This will be greater or less than for an empty solenoid, depending on whether the material is paramagnetic or diamagnetic.

Check the bound surface current:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m (\mathbf{H} \times \hat{\mathbf{n}}) = \chi_m n I \hat{\phi}$$

It will be in the same direction (paramagnetic), or opposite direction (diamagnetic) to the free current, I .

Ferromagnetism

Ferromagnets are strong and nonlinear.

In a ferromagnet, the dipoles like to all point in the same direction, and their mutual magnetic fields are enough to sustain each other without an external field – that is, the fields produced by all the other dipoles are enough to align any given dipole; this is the key.

Such materials consist of crystal domains, in which virtually all of the unpaired electron spins are aligned. Each contains billions of dipoles. However, a ‘normal’ piece of macroscopic media will have very many such domains, which are randomly oriented with respect to one another, giving little or no overall magnetization.

If we place such a material within a sufficiently strong external field, it will re-orient the most ‘vulnerable’ of the electron dipoles, those on the domain boundaries which are being twisted in different directions. The net effect of the external field is to *move the domain boundaries*. If this field is strong enough, one domain takes over the entire material, and the iron is said to “saturate”. It is now a ‘permanent’ magnet.

If we now reduce the applied field (or current), there is a *partial* return to randomness: see fig. 6.28. If we were to apply a strong enough field in the opposite

direction, all the dipoles would reverse, and so would the magnetization. (This is called “hysteresis”).

Note that, with an iron core, a small current leads to a huge field, compared with the case of no core. For example, if we use a coil of n turns per metre, and a current I , then $H = nI$, so H is proportional to I .

Further, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. But now M is very much bigger than H , so basically \mathbf{B} is proportional to \mathbf{M} : see fig. 6.29.

Above a temperature, known as the Curie point, thermal agitation is just too strong, and the mutual reinforcement is no longer enough: iron becomes paramagnetic, not ferromagnetic. This temperature is precise (770 C for iron), and is an example of a phase transition.